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Matrix perturbation theory for M-theory on a pp-wave

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Abstract: In this paper, we study the matrix model proposed by Berenstein, Maldacena, and Nastase to describe M-theory on the maximally supersymmetric pp-wave. We show that the model may be derived directly as a discretized theory of supermembranes in the pp-wave background, or alternatively, from the dynamics of D0-branes in type-IIA string theory. We consider expanding the model about each of its classical supersymmetric vacua and note that for large values of the mass parameter μ, interaction terms are suppressed by powers of μ^{-1}, so that the model may be studied in perturbation theory. We compute the exact spectrum about each of the vacua in the large-μ limit and find the complete (infinite) set of BPS states, which includes states preserving 2, 4, 6, 8, or 16 supercharges. Through explicit perturbative calculations, we then determine the effective coupling that controls the perturbation expansion for large μ and estimate the range of parameters and energies for which perturbation theory is valid.

Keywords: M-Theory, M(atrix) Theories, Penrose limit and pp-wave background.

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1. Introduction

At the present time, the most promising candidate for a microscopic definition of eleven-dimensional M-theory is the large-N limit of the quantum mechanics obtained from dimensionally reducing ten-dimensional U(N) SYM theory to 0+1 dimensions, with hamiltonians

$$H_0 = R\ \mathrm{Tr}\left(\frac{1}{2}\Pi_A^2 - \frac{1}{4}[X_A, X_B]^2 - \frac{1}{2}\Psi^\top\gamma^A[X_A,\Psi]\right), \qquad A, B = 1, \ldots, 9\,. \tag{1.1}$$

The initial evidence for this suggestion came from the idea that the quantum completion of eleven-dimensional supergravity should be realized as a theory of quantum supermembranes just as supergravity in ten dimensions is given its quantum completion via superstrings. Unlike the superstring case, the worldvolume supermembrane theory is an interacting theory and the theory contains no tunable coupling constant; however, de Wit, Hoppe, and Nicolai [1] showed that in the light cone gauge, the supermembrane action can be regularized to give the quantum mechanics (1.1), providing a possible avenue for directly studying the quantum supermembrane. It was soon realized that in contrast to the discrete spectrum of states obtained for a quantum superstring, the spectrum of the matrix model (1.1) is continuous, and at the time this appeared to be a rather troubling feature of the quantum supermembrane.

The matrix model was given new life following the realization that a quantum completion of eleven-dimensional supergravity should arise directly from string theory as the strong coupling limit of type-IIA strings. Based on this connection, Banks, Fischler, Shenker, and Susskind (BFSS) [2] argued that M-theory in the "infinite momentum frame" should be described by the limit of a large number of D0-branes in type-IIA string theory at low energies and weak coupling. Remarkably, this system is described again by the quantum mechanics (1.1), so essentially the same conclusion is reached from a very different approach. Happily, the D0-brane point of view provided a very natural explanation for the continuous spectrum of the matrix model: the theory should be interpreted as a second quantized theory containing both single and multiparticle states, rather than a first quantized theory. This point of view also provided a precise interpretation for the finite N matrix model, as the DLCQ of M-theory with N units of momentum in the compact direction [3].

Since the work of BFSS, matrix theory has passed numerous tests (for a recent review, see [4]), however an honest quantum mechanical study of the model seems extremely difficult, due to the flat directions in the potential (resulting in the continuous spectrum mentioned earlier), the difficulty of distinguishing single-particle and multi-particle states, the lack of a tunable coupling constant (interactions are of order one), and the large number of degrees of freedom in the large-N limit. Indeed, even determining the normalizable ground state wavefunction for the case $N = 2$ (proven to exist in [5]) is an unsolved and notoriously difficult problem.

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Recently [6], it has been shown by Berenstein, Maldacena, and Nastase (BMN) that the matrix model (1.1) is actually part of a one-parameter family of matrix models with 32 supercharges given by

$$H = H_0 + \frac{R}{2}\,\mathrm{Tr}\left(\sum_{i=1}^{3}\left(\frac{\mu}{3R}\right)^2 X_i^2 + \sum_{a=4}^{9}\left(\frac{\mu}{6R}\right)^2 X_a^2 + \right.$$
$$\left. + i\frac{\mu}{4R}\Psi^\top\gamma^{123}\Psi + i\frac{2\mu}{3R}\epsilon^{ijk}X_iX_jX_k\right). \quad (1.2)$$

The matrix model for general μ was proposed by BMN to give a description of the DLCQ of M-theory on the maximally supersymmetric pp-wave background [7]–[10] of eleven-dimensional supergravity[1]

$$ds^2 = -2dx^+dx^- + \sum_{A=1}^{9} dx^A dx^A - \left(\sum_{i=1}^{3}\frac{\mu^2}{9}x^ix^i + \sum_{a=4}^{9}\frac{\mu^2}{36}x^ax^a\right)dx^+dx^+ ,$$
$$F_{123+} = \mu . \quad (1.3)$$

In the large-N limit with fixed $p^+ = N/R$ and fixed μ, this matrix model should then describe M-theory on the pp-wave background in a sector with boost-invariant momentum parameter μp^+.

As pointed out in [6], the pp-wave matrix model for non-zero μ avoids a number of difficulties associated with the $\mu = 0$ case. Firstly, the quadratic terms in the potential remove all flat directions, and yield a potential that becomes infinite in all directions. This implies a discrete spectrum for the massive matrix model and, as shown in [6], an isolated set of classical supersymmetric vacua. These vacua may be associated with all possible ways of dividing up the momentum into different numbers of membranes, from a single membrane carrying N units of momenta to N membranes each carrying a single unit of momentum. Since these vacua are separated by potential barriers, we see that the massive matrix model (at low energies) also has a natural distinction between single and multi-membrane states.

In this paper, we study the massive matrix model and explore the consequences of these and a further simplifying feature: that the existence of a tunable parameter in the matrix model opens up the possibility of a weakly coupled regime which can be studied in perturbation theory.

We now provide a concise summary of our results as we outline the remainder of the paper.

Before proceeding to study the model we first clarify its origins.[2] In section 2 we show that the massive matrix model arises directly as the regularized theory of light-cone gauge supermembranes in the maximally supersymmetric pp-wave background. In section 3, we note that the model may also be derived from the D0-brane perspective, although here one has to assume that the weak background D0-brane action continues to be valid even in a region of high curvature, perhaps a consequence of supersymmetry.

[1] We have used conventions in which $x^\pm = (t \pm x)/\sqrt{2}$, different from those in BMN.

[2] The derivation provided in [6] was somewhat indirect, showing that the $N = 1$ model matches the action for a superparticle in the pp-wave background and then writing an extension of the theory to general N consistent with supersymmetry.

In section 4, we proceed with a direct study of the massive matrix model. We begin by reviewing the action and its symmetries and identify explicit matrix theory expressions for the generators of the superalgebra, verifying that the (anti)commutation relations are satisfied correctly. We note that the appearance of rotation generators along with the hamiltonian in the anticommutator of supersymmetry generators allows BPS states carrying angular momentum.[3] After these generalities, we review the classical vacua of the theory, collections of fuzzy spheres corresponding to sets of membrane giant gravitons [11, 12] carrying various fractions of the total light-cone momentum, p^+. We expand the matrix model about each of these classical vacua and note that in each case we obtain a quadratic theory with interactions suppressed by a coupling $(R/\mu)^{3/2}$. We conclude that for fixed values of the other parameters, the model may be studied in perturbation theory for sufficiently large μ.

In section 5, we consider the theory in the $\mu \to \infty$. Here, the different vacua become superselection sectors separated by an infinite potential barrier and interactions vanish. For each vacuum, we explicitly diagonalize the quadratic action, obtaining the precise oscillator masses and thus the exact spectrum of the theory. We find that the oscillator masses are proportional to μ and completely independent of N. In fact, N plays the role of an ultraviolet cutoff leaving a finite tower of oscillator masses that becomes an infinite tower for $N \to \infty$. This fits very well with the regularized supermembrane picture, and in particular, we may interpret the large-N limit of the spectrum obtained from the single fuzzy sphere vacuum as the quadratic spectrum of a single supermembrane in a large-μp^+ limit of the pp-wave background. From the exact spectrum, we are able to explicitly identify all BPS states of the theory and find infinite towers of states preserving 2, 4, 6, and 8 supercharges in addition to the 1/2 BPS vacuum states.

In section 6, we move away from the $\mu = \infty$ limit and investigate perturbation theory about the quadratic actions. We expand the interaction hamiltonian in terms of the canonically normalized oscillators, and find that the parameter suppressing the interaction terms is different for different vacua. In particular, the coupling for the N-membrane vacuum remains $g = (R/\mu)^{3/2}$ while the coupling for the single-membrane vacuum is reduced to $g = (R/(\mu N))^{3/2}$. This latter quantity, equals to $r^{-3/2}$ where r is the radius of the membrane sphere, remains finite in the M- theory limit of fixed μ and fixed $p^+ = N/R$, giving the coupling constant for the continuum field theory living on the single spherical supermembrane.

Since it is often the case that bare couplings become modified to some effective coupling in theories with many degrees of freedom, we then compute explicitly the energy shifts for certain non-BPS excited states of the single-membrane and N-membrane vacua. In both cases, we find that the leading corrections to the energy are suppressed by an effective coupling $\lambda = g^2 K$ relative to the energy in the free theory, where g is the bare coupling for the corresponding vacuum and K is the multiplicity of the representation (number of coincident branes). Thus, the leading perturbative correction in the irreducible vacuum is small when $g^2 = 1/(\mu p^+)^3 \ll 1$ while a small leading correction for the $X = 0$ vacuum requires a much stronger condition, $N^4/(\mu p^+)^3 \ll 1$.

[3] An example of such a state has been given in [24].

For fixed values of the parameters, we then estimate the energy scale at which perturbation theory will break down (i.e. the interaction Hamiltonian becomes of the same order of magnitude as the quadratic piece). For generic directions away from the single-membrane vacuum, we find that interactions become important at an energy $E/\mu \sim 1/g^2 = (\mu N/R)^3$ but there are certain directions (corresponding to the large membrane shrinking and a small membrane growing at the origin) for which interactions become important at $E/\mu \sim 1/(g^2 N) = \frac{1}{N}(\mu N/R)^3$. We expect that a valid perturbative regime will certainly exist when this energy is significantly greater than the energies of the lowest excited states, $g^2 N \ll 1$, as depicted in figure 1. On the other hand, our direct perturbative calculations indicate small perturbative corrections for low energy states under the much weaker condition $g^2 \ll 1$.

In section 7, we discuss the implications of our results for the large-N limit of the model. Our calculation of the leading energy correction indicates that perturbation theory near the single membrane vacuum may be valid even in the large N limit as long as $\mu p^+ \gg 1$. On the other hand, the more conservative estimate based on the height of the barrier between vacua suggests that we always exit the perturbative regime in the large-N limit defining M-theory (as in figure 1). In any case, our results explicitly demonstrate that in this limit, the theory develops finite zero-energy flat directions corresponding to decreasing the radius of any membrane while increasing the radius of any other membrane (or nucleating a new spherical membrane at the origin). This indicates the existence of massless singular BPS domain walls, or tensionless strings, in the supermembrane world-volume field theory.

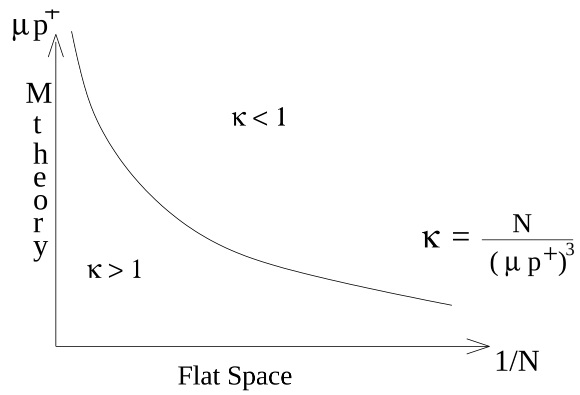

Figure 1: Regime of validity for perturbation theory (conservative).

Finally, we offer some concluding remarks in section 8 and technical results in a set of appendices.

2. The matrix model from supermembranes

It is well known that the usual Matrix theory action can derived as a discretized version of the light-cone gauge supermembrane action in flat space, where Poisson brackets on the membrane worldvolume are replaced with matrix commutators [1]. In other words, the invariance of the supermembrane action under area preserving diffeomorphisms is translated to the $\mathrm{U}(N), N \to \infty$ gauge invariance of the dimensionally reduced SYM theory which defines the matrix model.

In this section we show that the same story applies in the case of the pp-wave background. That is, a discretized version of the light-cone supermembrane action on the

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maximally supersymmetric eleven-dimensional pp-wave background exactly reproduces the matrix model proposed by BMN.

To start with, we borrow the supermembrane action from [13]:

$$S[Z(\sigma)] = \int d^3\sigma \, \sqrt{-G(Z(\sigma))} + \int C \,, \tag{2.1}$$

where $Z^M(\sigma) = X^\mu(\sigma), \theta^\alpha(\sigma)$ are the superspace embedding coordinates and $\sigma^i, i = 0, 1, 2$ are the world-volume membrane coordinates. Here,

$$G(Z(\sigma)) = \det(\Pi_i^\mu \Pi_j^\nu G_{\mu\nu}) \,, \tag{2.2}$$

where $\Pi_j^M = \partial Z^N / \partial\sigma^j E_N^M$ is the pull back of the supervielbein to the membrane world-volume and $G_{\mu\nu}$ is the background metric, which in our case is the pp-wave metric (1.3). The three form C field is

$$C = \frac{1}{6} e^r \wedge e^s \wedge e^t \; C_{\rm rst} + C_F \,, \tag{2.3}$$

where $e^r = e^r_\mu dX^\mu$ and e^r_μ are the vielbeins. The C_F term, which denotes the fermionic contributions to the C superfield, will be presented explicitly in section 2.2.

We begin with the bosonic terms and then consider the fermionic terms in the supermembrane action.

2.1 Bosonic terms

In order to obtain the matrix model from the membrane action one should fix the light-cone gauge. For that it will be convenient to use the coordinates (X^+, X^-, X^A) with $A = 1, \ldots, 9$. The light-cone condition is then $X^+ = \tau$, while the other worldvolume coordinates are denoted by σ^r, $r = 1, 2$. One should note that to fix the light-cone gauge consistently we should also impose the following constraint on the phase-space variables

$$P_A \partial_r X^A + P_- \partial_r X^- = 0 \,, \tag{2.4}$$

where P_A, P_- are the momentum conjugate to X^A and X^-, respectively.

Upon fixing the light-cone gauge, the bosonic part of the total hamiltonian becomes [13]

$$\begin{aligned} H_{\rm bos} = \int d^2\sigma \Bigg(& \frac{G_{+-}}{P_- - C_-} \left[\frac{1}{2} (P_A - C_A - \frac{P_- - C_-}{G_{+-}} G_{A+})^2 + \frac{1}{4} (\epsilon^{rs} \partial_r X^A \partial_s X^B)^2 \right] - \\ & - \frac{P_- - C_-}{2G_{+-}} G_{++} - C_+ + \frac{1}{P_- - C_-} \left[\epsilon^{rs} \partial_r X^A \partial_s X^B P_A C_{+-B} + C_- C_{+-} \right] \Bigg) \,, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} C_A &= -\epsilon^{rs} \partial_r X^- \partial_s X^B C_{-AB} + \frac{1}{2} \epsilon^{rs} \partial_r X^B \partial_s X^C C_{ABC} \,, \\ C_\pm &= \frac{1}{2} \epsilon^{rs} \partial_r X^A \partial_s X^B C_{\pm AB} \,, \\ C_{+-} &= \epsilon^{rs} \partial_r X^- \partial_s X^A C_{+-A} \,. \end{aligned} \tag{2.6}$$

Now let us concentrate on the pp-wave background, where

$$
\begin{aligned}
G_{--} &= G_{A-} = 0\,, \qquad G_{+-} = -1\,, \qquad G_{AB} = \delta_{AB}\,,\\
G_{++} &= -F^2 \equiv -\left(\frac{\mu^2}{9}\sum_{i=1}^{3} X_i^2 + \frac{\mu^2}{36}\sum_{a=1}^{6} X_a^2\right),
\end{aligned}
\tag{2.7}
$$

and

$$
C_- = C_{+-} = C_A = 0\,, \qquad C_+ = \frac{1}{2}\epsilon^{rs}\partial_r X^A \partial_s X^B C_{+AB} = \frac{\mu}{3!}\epsilon_{ijk}\epsilon^{rs}\partial_r X^i \partial_s X^j X^k\,. \tag{2.8}
$$

In this background the light-cone hamiltonian above simplifies significantly and we are left with

$$
\begin{aligned}
H_{\rm bos} = \frac{1}{p^+}\int d^2\sigma \Bigg[&\frac{1}{2}P_A^2 + \frac{1}{4}\{X^i, X^j\}^2 + \frac{1}{4}\{X^a, X^b\}^2 + \frac{1}{2}\{X^i, X^a\}^2 + \\
&+ \frac{1}{2}\left(\left(\frac{\mu p^+}{3}\right)^2 (X^i)^2 + \left(\frac{\mu p^+}{6}\right)^2 (X^a)^2\right) - \frac{\mu p^+}{6}\epsilon^{ijk}\{X^i, X^j\}X^k\Bigg]\,,
\end{aligned}
\tag{2.9}
$$

where $i, j = 1, 2, 3$, $a, b = 1, 2, \ldots, 6$, $p^+ = -P_-$ and

$$
\epsilon^{rs}\partial_r X^A \partial_s X^B = \{X^A, X^B\}\,.
$$

2.2 Fermionic terms

To get the fermionic terms we have to study the supervielbein and the tensor field backgrounds to all orders in θ. For coset spaces (of which the pp-wave is an example), the exact expressions for the supervielbein E_N^B,[4] in terms of the component worldvolume and background fields have been worked out in [14], and are given by[5]

$$
E = D\theta + \sum_{n=1}^{16} \frac{1}{(2n+1)!}\mathcal{M}^{2n} D\theta\,, \tag{2.10}
$$

$$
E^r = e^r + \bar\theta\Gamma^r D\theta + 2\sum_{n=1}^{15} \frac{1}{(2n+2)!}\bar\theta\Gamma^r \mathcal{M}^{2n} D\theta\,. \tag{2.11}
$$

In these expressions,

$$
\begin{aligned}
e^r &= e^r_\mu dX^\mu\,,\\
\omega^{rs} &= \omega^{rs}_\mu dX^\mu\,,\\
D\theta &\equiv d\theta + e^r T_r^{\rm stuv}\theta F_{\rm stuv} - \frac{1}{4}\omega^{rs}\Gamma_{rs}\theta\,,
\end{aligned}
\tag{2.12}
$$

where ω^{rs} is the spin connections, $F_{\rm stuv}$ is the field strength of the three form C and

$$
T_r{}^{\rm stuv} = \frac{1}{2!3!4!}\left(\Gamma_r{}^{\rm stuv} - 8\delta_r{}^{[s}\Gamma^{\rm tuv]}\right). \tag{2.13}
$$

[4]The $B = (r, a)$ indices are tangent space indices and the curved indices are $N = (\mu, \alpha)$, where the first entries are bosonic and the second entries are fermionic.

[5]The coset space approach was also used in [15, 16] to derive the action for light cone type-IIB string theory in the pp-wave background.

Finally, the matrix $\mathcal{M}^2$ is defined as

$$(\mathcal{M}^2)^a_b = 2(T_r^{\rm stuv}\theta)^a F_{\rm stuv}(\bar\theta\Gamma^r)_b - \frac{1}{288}(\Gamma_{rs}\theta)^a \left(\bar\theta[\Gamma^{\rm rstuvw}F_{\rm tuvw} + 24\Gamma_{tu}F^{rstu}]\right)_b . \qquad (2.14)$$

In the pp-wave background,

$$e^-_{\ -} = e^+_{\ +} = 1\,, \qquad e^A_{\ B} = \delta^A_{\ B}\,, \qquad e^-_{\ +} = \frac{F^2}{2}\,, \qquad (2.15)$$

and all the other components are zero. Then, the only non-vanishing component of the spin connection is

$$\omega_+^{\ -A} = \frac{1}{2}\partial_A F^2\,. \qquad (2.16)$$

To work out the explict form of the fermionic fields we should choose a specific representation for Γ matrices. The 32×32 Γ matrices can be written as

$$\Gamma^+ = \sqrt{2}\begin{pmatrix} 0 & 0 \\ i\mathbf{1_{16}} & 0 \end{pmatrix}, \qquad \Gamma^- = \sqrt{2}\begin{pmatrix} 0 & i\mathbf{1_{16}} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^A = \begin{pmatrix} \gamma^A & 0 \\ 0 & -\gamma^A \end{pmatrix}, \qquad (2.17)$$

where the γ^A are 16×16 euclidean SO(9) gamma matrices.[6] We can also use κ-symmetry to fix the (light-cone) gauge:

$$\Gamma^+\theta = 0\,. \qquad (2.18)$$

In this gauge we have the advantage of significant simplifications such as

$$\bar\theta\Gamma^+ = \bar\theta\Gamma^{+-}\partial\theta = \bar\theta\Gamma^{+A}\partial\theta = \bar\theta\Gamma^{+AB}\partial\theta = \bar\theta\Gamma^A\partial\theta = \bar\theta\Gamma^{AB}\partial\theta = 0, \dots\,. \qquad (2.19)$$

One can solve the gauge condition (2.18) explicitly:

$$\theta = \frac{1}{2^{1/4}}\begin{pmatrix} 0 \\ \Psi \end{pmatrix}, \qquad \bar\theta = \frac{1}{2^{1/4}}\begin{pmatrix} -\Psi^\top & 0 \end{pmatrix}, \qquad (2.20)$$

where Ψ is a 16-dimensional (Majorana) spinor. The normalization for Ψ have been chosen so that

$$i\bar\theta\Gamma^-\theta = \Psi^\top\Psi\,.$$

To work out the action explicitly, we first note that for the pp-wave case, as discussed earlier, the only non-trivial background field component appearing in $\mathcal{M}^2$ is $F_{+123} = -F^{-123}$. Therefore, using the gauge condition (2.18) one can show that

$$\mathcal{M}^2 = 0\,. \qquad (2.21)$$

That is, the only non-vanishing fermionic contributions come from the expansion of the supervielbeins to the leading order in θ.

Now let us focus on the fermionic contributions coming from the $\sqrt{-G}$ terms in the action. For these, we need to work out Π_j^r's:

$$\Pi_i^r = E^r_\nu\partial_i X^\nu + E^r_\alpha\partial_i\theta^\alpha\,. \qquad (2.22)$$

[6]Note that $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^{10})$, and $\bar\theta = i\theta^\dagger\Gamma^0$.

Using the eq. (2.10), we have

$$E_\nu^r = e_\nu^r + \bar\theta\Gamma^r D_\nu\theta\,, \qquad E_\alpha^r = \bar\theta\Gamma^r D_\alpha\theta\,, \tag{2.23}$$

where

$$\bar\theta\Gamma^r D_\nu\theta = \frac{i\mu}{4}\delta_\nu^+\delta_-^r\ \Psi^\top\gamma^{123}\Psi\,,$$

and $E_\alpha^r = -(\bar\theta\Gamma^r)_\alpha$. Putting this all together we obtain

$$\begin{aligned}\Pi_i^- &= \partial_i X^- + \partial_i X^+\left(\frac{F^2}{2} + \frac{i\mu}{4}\Psi^\top\gamma^{123}\Psi\right) + i\Psi^\top\partial_i\Psi\,,\\ \Pi_i^+ &= \partial_i X^+\,, \qquad \Pi_i^A = \partial_i X^A\,.\end{aligned}\tag{2.24}$$

We can see that all the effects of the pp-wave background appear in the $\partial_i X^+$ factor of Π_i^-. Hence, in the membrane hamiltonian (2.5) G_{++} should be shifted to

$$G_{++} + \frac{i\mu}{2}\Psi^\top\gamma^{123}\Psi\ ,$$

which leads to a "mass term" for the fermions. Note that the other fermionic term $i\Psi^\top\partial_i\Psi$ will not contribute to the light-cone hamiltonian (because it involves first order time derivatives).

The only remaining terms to be considered are the fermionic contributions to the C term, C_F [14]:

$$C_F = -\int_0^1 tdt\ \bar\theta\Gamma_{rs}\ E(t)\wedge E^r(t)\wedge E^s(t)\,, \tag{2.25}$$

where t is an auxiliary parameter and $E(t)$, $E^r(t)$ are obtained from (2.10) by the shift $\theta\to t\theta$. As we discussed, using the gauge fixing condition (2.18), $\mathcal{M}^2$ is zero and hence

$$C_F = -\int_0^1 t^2dt\ \bar\theta\Gamma_{rs}D\theta\wedge(e^r + t^2\bar\theta\Gamma^r D\theta)\wedge(e^s + t^2\bar\theta\Gamma^s D\theta)\,.$$

In the gauge (2.18), $\bar\theta\Gamma_{rs}D\theta$ is zero unless $\{rs\} = \{+A\}$, while $\bar\theta\Gamma^r D\theta$ is zero if r is $+$. Therefore the only surviving term is

$$C_F = -\int_0^1 t^2dt\ \bar\theta\Gamma_{rs}D\theta\wedge e^r\wedge e^s = \frac{1}{3}\epsilon^{ijk}\bar\theta\Gamma^{-A}\partial_i X^+\partial_j\theta\partial_k X^A = \frac{i}{2}\Psi^\top\gamma^A\{X^A,\Psi\}\,. \tag{2.26}$$

In summary, gathering together all fermionic and bosonic terms, we have shown that the total light-cone gauge hamiltonian for membranes in the maximally supersymmetric pp-wave background, with properly fixed κ-symmetry, is:

$$\begin{aligned}H_{\rm total} = \int d^2\sigma\Bigg\{\frac{1}{p^+}\Bigg[&\frac{1}{2}P_A^2 + \frac{1}{4}\{X^A,X^B\}^2 + \frac{1}{2}\left(\left(\frac{\mu p^+}{3}\right)^2(X^i)^2 + \left(\frac{\mu p^+}{6}\right)^2(X^a)^2\right) -\\ &-\frac{\mu p^+}{6}\epsilon^{ijk}\{X^i,X^j\}X^k\Bigg] - \frac{i}{2}\Psi^\top\gamma^A\{X^A,\Psi\} - \frac{i}{4}\mu\Psi^\top\gamma^{123}\Psi\Bigg\}\,.\end{aligned}\tag{2.27}$$

The matrix model hamiltonian proposed by BMN now follows immediately by applying the regularization prescription of [1, 21] , in which we replace functions on the membrane worldvolume by hermitean $N \times N$ matrices, integrals with traces, and Poisson-brackets with commutators, i.e.

$$\begin{aligned} X^A(\sigma) &\longleftrightarrow \frac{1}{N} X^A\,, \qquad \Psi(\sigma) \longleftrightarrow \Psi\,, \\ p^+ \int d^2\sigma &\longleftrightarrow \frac{1}{R}\,\mathrm{Tr}\,, \qquad \{\ ,\ \}_{PB} \longleftrightarrow -i\,[\ ,\ ]\,. \end{aligned}$$

In the next section, we show explicitly how the matrix model may also be obtained from the D0-brane perspective.

3. The matrix model from D0-brane dynamics

In the previous section, we have seen that the matrix model proposed in [6] arises directly as a regularized version of the light-cone supermembrane action in the maximally supersymmetric pp-wave background of eleven-dimensional supergravity, just as ordinary Matrix theory arises from the flat space supermembrane.

In the usual case, an alternate approach to deriving the matrix model as a description of M-theory is by showing an equivalence between the DLCQ of M-theory with N units of lightlike momentum [3] and a limit of type-IIA string theory with N D0-branes in which the only remaining degrees of freedom are the lightest open strings between the D0-branes [2, 17, 18]. In this section, we will try to obtain the matrix model of BMN directly from this D0-brane perspective.

3.1 Relation to type-IIA string theory

In the usual arguments [17, 18], DLCQ M-theory with N units of momentum is obtained (or defined) by taking a compactification of M-theory on a spacelike circle of radius R_s, performing a boost with parameter

$$\gamma = \sqrt{\frac{R^2}{2R_s} + 1} \tag{3.1}$$

and taking the limit $R_s \to 0$ where the boost parameter becomes infinite leaving a lightlike circle of fixed radius R. The original frame, with spacelike circle of radius R_s, is equivalent to type-IIA string theory with N D0-branes in a limit $g_s \to 0$, $\alpha' \to 0$ in which only the low energy D0-brane degrees of freedom remain. In order that the energies of the corresponding D0-brane states (minus the diverging constant piece N/R_s) match with the light cone energies in the DLCQ description, we must also rescale all dimensionful quantities in the D0-brane theory by $m \to (R/R_s)m$ before taking the limit.

In the present case, we begin with a non-flat background for DLCQ M-theory, namely the pp-wave metric. Applying the same arguments, we should find that the matrix model appropriate to this background is that describing D0-branes in a type-IIA background obtained by (un)boosting and compactifying the eleven-dimensional pp-wave background.

To determine the relevant IIA background in the present case (where the background fields are not necessarily weak), note that under a boost to the speed v in $x^{\pm}$ plane,

$$x^+ \longrightarrow \sqrt{\frac{1-v}{1+v}}x^+ , \qquad x^- \longrightarrow \sqrt{\frac{1+v}{1-v}}x^- .$$

Hence, the pp-wave metric is boost covariant, and the effect of the boost is simply to rescale μ. In fact, up to a factor of $\sqrt{2}$, this rescaling is later removed when we rescale dimensionful parameters to make light-cone energies match with the energies of D0-brane states, so we will take both into account at once simply by replacing $\mu \to \mu/\sqrt{2}$. More details of the boost and rescaling may be found in [19], where this procedure was carried out in detail for an arbitrary weak background.

Taking into account the rescaling, the metric for M-theory in the "unboosted" frame, which we should compactify on a spacelike circle to obtain the desired type-IIA metric, is

$$ds^2 = -\left(1+\frac{F^2}{4}\right)dt^2 + \left(1-\frac{F^2}{4}\right)(dx^{10})^2 - \frac{F^2}{2}dtdx^{10} + \sum_{i=1}^{3} dx_i dx_i + \sum_{a=4}^{9} dx_a dx_a \, , \quad (3.2)$$

where F^2 is defined in (2.7) and the four form field is

$$F_{123(10)} = -F_{0123} = \frac{1}{2}\mu \, . \quad (3.3)$$

Compactifying along x^{10}, the corresponding IIA background (in string frame) is

$$\begin{aligned} ds^2_{10} &= g_{00}dt^2 + g_{AB}dx^A dx^B \, , \\ g_{00} &= -e^{-2\phi/3} \, , \qquad g_{AB} = \delta_{AB}e^{2\phi/3} \, , \qquad A,B = 1,\ldots,9 \end{aligned} \quad (3.4)$$

and

$$e^{4\phi/3} = 1 - \frac{1}{4}F^2 \, , \qquad C_0 = \frac{G_{010}}{G_{10,10}} = -\frac{F^2}{4-F^2} \, , \quad (3.5)$$

$$F_{0123} = -\frac{\mu}{2} \, , \qquad H_{123} = \frac{\mu}{2} \, , \quad (3.6)$$

where C_0 is the (time-component) of the RR-one form potential and H_{123} and F_{0123} are the NSNS and RR-three form field strengths, respectively.

Although the eleven-dimensional pp-wave metric that we started with has zero scalar curvature (and its only non-vanishing component of Ricci scalar is $R_{++} = \frac{1}{2}\mu^2$), the ten-dimensional metric has a non-zero curvature. This can be understood noting that the effective compactification radius ($e^{2\phi/3}$) has a non-trivial x-dependence. By explicit calculation one can work out the scalar curvature of the metric (3.4):

$$\mathcal{R} = -\frac{\mu^2}{8}\left(1-\frac{F^2}{4}\right)^{-3/2}\left[a - b\,\mu^2\sum_{i=1}^{3}x_i x_i - c\,\mu^2\sum_{a=4}^{9}x_a x_a\right] , \quad (3.7)$$

where a, b, c are positive constants.

We can trust the above SUGRA solution only when the curvature is small, i.e. when $F^2 \ll 4$. In fact, we see that at $F^2 = 4$, the background becomes singular, and is ill-defined beyond this point. From the eleven-dimensional point of view, the reason for this bad behavior is that the x^{10} direction becomes timelike beyond this point, so clearly, compactification does not make sense. Nevertheless, we may hope that in the region for which the type-IIA background is well behaved, the action describing D0-branes in this background may be a valid description of the regime in which the DLCQ M-theory degrees of freedom stay within the weakly coupled region of the pp-wave background.

In this weakly curved limit, the IIA background fields can be expanded in powers of F^2. Up to the first order in F^2 we find

$$
\begin{aligned}
e^{4\phi/3} &= 1 - \frac{F^2}{4} \Longrightarrow \phi \sim -\frac{3}{16}F^2\,, \qquad C_0 = -\frac{F^2}{4-F^2} \sim -\frac{F^2}{4}\,,\\
g_{00} &= -e^{-2\phi/3} \sim -1 - \frac{1}{8}F^2\,, \qquad g_{AB} = e^{2\phi/3}\delta_{AB} \sim \delta_{AB} - \frac{1}{8}F^2\delta_{ij}\,.
\end{aligned}
\tag{3.8}
$$

We proceed in the next subsection to determine the action for D0-branes in this background.

3.2 D0-brane dynamics on compactified reduced pp-waves

We have seen that it is only possible to relate the eleven-dimensional pp-wave metric to a reliable type-IIA supergravity background in the region where $F^2 \ll 4$. In this weakly curved region, we may read off the action describing D0-branes in the background (3.8) from the results of [19], which provides the action for D0-branes at low energies and weak coupling in a general weak background.

In the notation of that paper, terms arising from the eleven-dimensional metric for our background are

$$
S_2 = \frac{1}{2}\partial_A\partial_B\phi I_\phi^{(AB)} + \frac{1}{4}\partial_A\partial_B h_{\mu\nu} I_h^{\mu\nu(AB)} + \frac{1}{2}\partial_A\partial_B C_0 I_0^{0(AB)}\,,
\tag{3.9}
$$

where, for example $I_\phi^{(AB)}$ is the AB moment of the current coupling to ϕ. Using the expressions for the currents I given in [19] and the background fields given in (3.8), we obtain

$$
S_2 = -\frac{1}{4}\partial_A\partial_B F^2(T^{++(AB)} + \text{higher orders})\,,
$$

where

$$
T^{++(AB)} = \frac{1}{R}\,\mathrm{Tr}(X^A X^B)
$$

and the higher order terms vanish in the $\alpha' \to 0$ limit we are taking. Plugging in our expression for F^2, we obtain exactly the bosonic mass terms in the BMN matrix model.

To obtain the terms arising from the eleven-dimensional three-form field, the required terms in the zero-brane action are

$$
S_3 = \partial_A B_{\mu\nu} I_s^{\mu\nu(A)} + \partial_A C_{\alpha\beta\gamma} I_2^{\alpha\beta\gamma(A)}\,.
\tag{3.10}
$$

To obtain the field strengths (3.6), we may choose potentials

$$
B_{ij} = \frac{\mu}{6}\epsilon_{ijk}x_k\,, \qquad C_{0ij} = \frac{\mu}{6}\epsilon_{ijk}x_k\,.
\tag{3.11}
$$

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Using these expressions and the expressions given for the currents in [19], we find

$$S_3 = \mu\epsilon_{ijk}(J^{+ij(k)} + \text{higher orders})\,,$$

where

$$J^{+ij(k)} = -\frac{1}{6R}\,\mathrm{Tr}\left(i[X^i,X^j]X^k + \frac{i}{8}\Psi^\top\gamma^{ijk}\Psi\right)$$

and again, the higher order terms vanish in the limit we are considering. Thus, the form-field background gives rise to both the fermion mass terms and the bosonic cubic interaction appearing in the BMN matrix model.

We have now seen that all terms in the BMN matrix model may be reproduced by considering D0-branes in an appropriate weak supergravity background.[7] From this point of view, it is remarkable that the model does not receive any corrections, since we have seen that the type-IIA background becomes strongly curved and unreliable as we move far from the origin and in fact fails to make sense for $F^2 \geq 4$.

4. Classical supersymmetric vacua

In this section, we begin a direct study of the matrix model. We first write down the action in a convenient form and discuss the symmetries of the theory and their consequences for the spectrum. We then recall the supersymmetric vacua of the model and expand the matrix theory action about each of these solutions. We find that near each vacuum, the theory is described by a quadratic hamiltonian with interactions suppressed by $1/\mu$.

4.1 Action and symmetries

It will be most convenient to write the action in a form that makes the SO(3) $\times$ SO(6) symmetries manifest, so we divide the scalars into X^i with $i = 1,2,3$ and X^a with $4 \leq a \leq 9$, and write the fermions as $\psi_{I\alpha}$ where I is a fundamental index of SU(4) $\sim$ SO(6) and α is a fundamental index of SU(2) $\sim$ SO(3). The relation between these fermions and the real 16 component spinor Ψ is detailed in appendix A.

With these conventions, the matrix model describing the DLCQ of M-theory in the maximally supersymmetric pp-wave background is given by[8]

$$\begin{aligned}
\mathcal{L} = {} & \mathrm{Tr}\left(\frac{1}{2R}D_0X^iD_0X^i + \frac{1}{2R}D_0X^aD_0X^a + i\psi^{\dagger I\alpha}D_0\psi_{I\alpha}\right) + \\
& + R\ \mathrm{Tr}\left(-\frac{1}{2}\left(\frac{\mu}{3R}\right)^2(X^i)^2 - \frac{1}{2}\left(\frac{\mu}{6R}\right)^2(X^a)^2 - \frac{\mu}{4R}\psi^{\dagger I\alpha}\psi_{I\alpha} - \frac{i\mu}{3R}\epsilon_{ijk}X^iX^jX^k - \right.\\
& \qquad - \psi^{\dagger I\alpha}\sigma^{i\ \beta}_{\alpha}[X^i,\psi_{I\beta}] + \frac{1}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha I}\mathrm{g}^a_{IJ}[X^a,\psi^{\dagger\beta J}] - \\
& \qquad \left. - \frac{1}{2}\epsilon^{\alpha\beta}\psi_{\alpha I}(\mathrm{g}^{a\dagger})^{IJ}[X^a,\psi_{\alpha J}] + \frac{1}{4}[X^i,X^j]^2 + \frac{1}{4}[X^a,X^b]^2 + \frac{1}{2}[X^i,X^a]^2\right).
\end{aligned} \tag{4.1}$$

[7] These results could also be obtained directly using the general results for matrix theory in an arbitrary weak background [20].

[8] Throughout this paper, we set $l_P = 1$, but we can restore l_P in any formulae using the fact that R and $1/\mu$ have dimensions of length.

Here, σ^i are the usual Pauli matrices and g^a_{IJ} relate the antisymmetric product of two SU(4) fundamentals to an SO(6) fundamental.

The new terms relative to the flat space matrix model appear in the second line. These include mass terms for each of the matrix theory variables, as well as a cubic interaction which favors matrix configurations carrying a dipole moment of membrane charge.

As for the case of flat space, the U(1) part of the theory (describing the center of mass degrees of freedom) decouples from the SU(N) part. In this case, the U(1) sector is described by a harmonic oscillator hamiltonian with bosonic oscillators in the SO(6) directions of mass $\mu/6$ and in the SO(3) directions of mass $\mu/3$, as well as 8 fermionic oscillators of mass $\mu/4$. Thus, unlike the flat space case, the different polarization states have different masses.

In appendix B, we review the full symmetry algebra of the matrix model and provide explicit expressions for the generators in terms of the matrix theory variables. The bosonic generators include the hamiltonian, rotation generators in the SO(3) and SO(6) directions, the central light-cone momentum $P^+ = N/R$, and generators a_i and a_a which are exactly the harmonic oscillator creation operators from the U(1) part of the theory. In addition, we have 32 fermionic symmetries. Sixteen of the generators are the 8 complex fermionic creation operators from the U(1) part of the theory,

$$q_{I\alpha} = \frac{1}{\sqrt{R}}\,\mathrm{Tr}(\psi_{I\alpha})$$

which generate the overall polarization states. The other 16 supersymmetries are generated by

$$\begin{aligned} Q_{I\alpha} = \sqrt{R}\,\mathrm{Tr}\Big(\Big(\Pi^a - i\frac{\mu}{6R}X^a\Big)\,\mathrm{g}^a_{IJ}\epsilon_{\alpha\beta}\psi^{\dagger J\beta} - \Big(\Pi^i + i\frac{\mu}{3R}X^i\Big)\,\sigma^{i\ \beta}_{\alpha}\psi_{I\beta} + \\ + \frac{1}{2}[X^i,X^j]\epsilon^{ijk}\sigma^{k\ \beta}_{\alpha}\psi_{I\beta} - \frac{i}{2}[X^a,X^b](\mathrm{g}^{ab})_I^{\ J}\psi_{J\alpha} + i[X^i,X^a]\sigma^i\mathrm{g}^a_{IJ}\epsilon_{\alpha\beta}\psi^{\dagger I\beta}\Big) \end{aligned}$$

which include new terms linear in X relative to the flat space model.

The supersymmetry algebra has a number of unusual features compared with the flat space case. First, the supercharges do not commute with the hamiltonian but rather obey commutation relations

$$[H, q_{I\alpha}] = -\frac{\mu}{4}q_{I\alpha}\,, \qquad [H, Q_{I\alpha}] = \frac{\mu}{12}Q_{I\alpha}\,.$$

Thus, different members of a multiplet of states generated by acting with supercharges will have different energies, although the energy differences will still be exactly determined by the symmetry algebra. In addition, we now have rotation generators appearing on the right-hand side of the $\{Q,Q\}$ commutation relation along with the hamiltonian,

$$\{Q^{\dagger I\alpha}, Q_{J\beta}\} = 2\delta^I_J\delta^\alpha_\beta H - \frac{\mu}{3}\epsilon^{ijk}\sigma^{k\alpha}_{\beta}\delta^I_J M^{ij} - \frac{i\mu}{6}\delta^\alpha_\beta(\mathrm{g}^{ab})_J^{\ I}M^{ab}\,.$$

Note that we still have $\{Q^{\dagger I\alpha}, Q_{I\alpha}\} \propto H$, so that a state is fully supersymmetric if and only if it has zero energy. On the other hand, unlike the ordinary matrix model, we now have the possibility of BPS states with non-zero energies carrying some angular momentum. We will see that 1/4, 3/16, 1/8, and 1/16 BPS states are possible in addition to the ground states which are 1/2 BPS (preserving 16 supercharges). In this case, the rotation generators are not central charges since they do not commute with the supersymmetry generators. As a result, the representation theory of this algebra is quite different from the usual case, and in particular the notion of BPS multiplets is altered. We will discuss the BPS states in more detail in section 5.5.

4.2 Classical supersymmetric solutions

We now recall the classical supersymmetric vacua of the matrix model given in [6]. To begin, we note that the bosonic potential may be written as a sum of positive definite terms,

$$V=\frac{R}{2}\ \mathrm{Tr}\left[\left(\frac{\mu}{3R}X^i+i\epsilon^{ijk}X^jX^k\right)^2+\frac{1}{2}(i[X^a,X^b])^2+(i[X^a,X^i])^2+\left(\frac{\mu}{6R}\right)^2(X^a)^2\right].$$

For supersymmetric solutions, each of the four terms must vanish independently. It is clear that for the last term to vanish, we must take $X^a=0$, while for the first term to vanish we must have

$$X^i=\frac{\mu}{3R}J^i\,,$$

where J^i form a representation of the SU(2) algebra,

$$[J^i,J^j]=i\epsilon^{ijk}J^k\,.$$

Thus, the classical supersymmetric solutions are labeled by the N-dimensional representations of SU(2), and the number is given by the number of partitions of N into sums of positive integers, as noted in [6].

Geometrically, a solution labeled by a partition $\{N_1,\ldots,N_k\}$ corresponds to a set of membrane fuzzy spheres with radii $r_i\sim N_i\mu/6R$. The spheres are stabilized through a competition between the quadratic and quartic terms in the potential, which make the spheres want to contract, and the cubic term which makes the spheres want to expand. The net effect is that any membrane in the pp-wave background expands to a preferred radius proportional to its light-cone momentum. Unlike the similar situation studied by Myers [11], the various solutions all have zero energy and are thus classically stable.

It is particularly interesting that the classical vacuum states each correspond to a definite number of membranes. Thus, at low energies, one of the difficult features of the usual matrix model, distinguishing between single and multi- membrane states, is absent.

4.3 Expansion about the classical vacua

We now consider expanding the action about its classical vacua, that is taking

$$X^i=\frac{\mu}{3R}J^i+Y^i \tag{4.2}$$

for a given choice of J.

It is convenient to rescale variables to remove all dependence on parameters from the quadratic action, so we take

$$Y^i \longrightarrow \sqrt{\frac{R}{\mu}} Y^i \,, \qquad X^a \longrightarrow \sqrt{\frac{R}{\mu}} X^a \,, \qquad t \longrightarrow \frac{1}{\mu} t \,. \tag{4.3}$$

Note that energies will be measured in units of μ from now on; we will restore this factor in the important formulae.

Plugging (4.2) into the action and performing the rescalings, we find that the resulting action (in the $A_0 = 0$ gauge) takes the form

$$S = S_2^Y + S_2^X + S_2^\psi + S_3 + S_4 \,,$$

where

$$\begin{aligned} S_2^Y &= \mathrm{Tr}\left(\frac{1}{2}\dot{Y}^i\dot{Y}^i - \frac{1}{2}\left(\frac{1}{3}\right)^2 (Y^i + i\epsilon^{ijk}[J^j, Y^k])^2 \right), \\ S_2^X &= \mathrm{Tr}\left(\frac{1}{2}(\dot{X}^a)^2 - \frac{1}{2}\left(\frac{1}{3}\right)^2 \left(\frac{1}{4}(X^a)^2 - [J^i, X^a]^2 \right) \right), \\ S_2^\psi &= \mathrm{Tr}\left(i\psi^\dagger\dot{\psi} - \frac{1}{4}\psi^\dagger\psi - \frac{1}{3}\sigma^i\psi^\dagger[J^i, \psi_\beta] \right) \end{aligned} \tag{4.4}$$

are quadratic actions for X, Y, and ψ, and

$$\begin{aligned} S_3 &= \left(\frac{R}{\mu}\right)^{3/2} \mathrm{Tr}\left(\frac{1}{3}[J^i, X^a][Y^i, X^a] + \frac{1}{3}[J^i, Y^j][Y^i, Y^j] - \frac{i}{3}\epsilon^{ijk}Y^iY^jY^k - \right. \\ &\qquad \left. - \psi^\dagger\sigma^i[Y^i, \psi] + \frac{1}{2}\epsilon\psi^\dagger \mathrm{g}^a[X^a, \psi^\dagger] - \frac{1}{2}\epsilon\psi(\mathrm{g}^\dagger)^a[X^a, \psi] \right), \\ S_4 &= \left(\frac{R}{\mu}\right)^3 \mathrm{Tr}\left(\frac{1}{4}[Y^i, Y^j]^2 + \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[Y^i, X^a]^2 \right) \end{aligned} \tag{4.5}$$

are the interaction terms.

We see that for large μ/R, the interaction terms are suppressed by powers of R/μ relative to the quadratic action, so we should be able to study the low energy excitations about these classical vacua using perturbation theory in R/μ. In particular in the $\mu/R \to \infty$ limit, the theory divides into superselection sectors labeled by N-dimensional representations of SU(2) and each section will be described by a set of bosonic and fermionic harmonic oscillators. Our next step will be to solve the quadratic theory explicitly in this limit before considering the effects of interactions in section 6.

5. Exact spectrum in the large-μ limit

In the limit of large μ for fixed N and R, the various classical vacua become separated by infinite energy barriers and may be treated as superselection sectors, each described by a quadratic hamiltonian. As a first step towards understanding the theory, we would now like to explicitly diagonalize the quadratic actions in order to find the complete spectrum in the limit of large μ.

5.1 Expansion in terms of matrix spherical harmonics

To proceed, we note that many of the terms in the quadratic action are written in terms of commutators with the J^i matrices,

$$[J^i, A] \,.$$

It will be convenient then to expand our fluctuation matrices in a basis which behaves simply under this commutator action. To do this, note that the set of $N \times N$ matrices forms an N^2-dimensional reducible representation of SU(2), with the SU(2) generators acting as above. It is not hard to see that this representation decomposes into irreducible representations as

$$N^2 = 1 \oplus 3 \oplus \cdots \oplus (2N-1) \,,$$

that is, integer spins up to $j = N-1$. We may therefore expand an $N \times N$ matrix as

$$A = \sum_{j=0}^{N-1} \sum_{m=-j}^{j} a_{jm} Y_{jm} \,,$$

where Y_{jm} transform in the irreducible spin j representation. For hermitean matrices, we must have the reality condition

$$a^*_{jm} = (-1)^m a_{j\ -m} \,. \tag{5.1}$$

Explicit expressions for these matrix spherical harmonics may be obtained by writing the usual spherical harmonics as symmetric polynomials in x_i with $x_1^2 + x_2^2 + x_3^2 = 1$ and replacing x's with Js (up to normalization). In this way, we find that the spin j representation may be written as products of j J^i matrices whose indices are contracted with symmetric, traceless tensors. A detailed discussion may be found in [21].

For example

$$\begin{aligned} Y_{00} &= 1 \,, \qquad Y_{1\ -1} = \sqrt{\frac{6}{(N^2-1)}} J^- \,, \\ Y_{10} &= \sqrt{\frac{12}{(N^2-1)}} J^3 \,, \qquad Y_{11} = -\sqrt{\frac{6}{(N^2-1)}} J^+ \,, \ldots \,, \end{aligned} \tag{5.2}$$

where $J^\pm = J^1 \pm iJ^2$. In general, we will choose normalizations so that

$$\mathrm{Tr}(Y^\dagger_{j'm'} Y_{jm}) = N \delta_{j'\,j} \delta_{m'\,m} \,. \tag{5.3}$$

With these definitions, we may use the usual properties of angular momentum generators,

$$\begin{aligned} [J_3, Y_{jm}] &= m Y_{jm} \,, \qquad [J^+, Y_{jm}] = \sqrt{(j-m)(j+1+m)} Y_{j\ m+1} \,, \\ [J^i, [J^i, Y_{jm}]] &= j(j+1) Y_{jm} \,, \qquad [J^-, Y_{jm}] = \sqrt{(j+m)(j+1-m)} Y_{j\ m-1} \,. \end{aligned} \tag{5.4}$$

We now proceed to diagonalize each of the quadratic actions above.

5.2 Spectrum of the irreducible vacuum

We begin by considering the single-membrane vacuum corresponding to the N-dimensional irreducible representation of SU(2).

Bosons in the SO(6) directions

From the action above, we see that the quadratic potential for X^a may be written as

$$V^X = \frac{1}{2}\frac{1}{3^2}\,\mathrm{Tr}\left(X^a\left(\frac{1}{4}X^a + [J^i,[J^i,X^a]]\right)\right).$$

In this form, it is obvious that the mass matrix is diagonal in the basis of irreducible representations of SU(2),

$$X^a = \frac{1}{\sqrt{N}}x^a_{jm}Y_{jm}$$

and the eigenvalues are

$$M^2 = \left(\frac{1}{3}\left(j+\frac{1}{2}\right)\right)^2$$

each with degeneracy 6. The quadratic lagrangian becomes

$$\mathcal{L} = \sum_{j=0}^{N-1}\frac{1}{2}|\dot{x}^a_{jm}|^2 - \frac{1}{2}\left(\frac{1}{3}\left(j+\frac{1}{2}\right)\right)^2|x^a_{jm}|^2\,,$$

where j runs from 0 to $N-1$ and m runs from $-j$ to j, as usual.

Bosons in the SO(3) directions

For the Y^i oscillators, we may rewrite the quadratic potential as

$$V^Y = \frac{1}{2}\left(\frac{1}{3}\right)^2\left(Y^i + i\epsilon^{ijk}[J^j,Y^k]\right)^2.$$

Thus, if we solve the eigenvalue problem

$$Y^i + i\epsilon^{ijk}[J^j,Y^k] = \lambda Y^i$$

the masses will be given as $M^2 = (1/3\lambda)^2$.[9]

To find the eigenvalues and eigenvectors, we write

$$Y^i = y^i_{jm}Y_{jm}\,.$$

Making this substitution and writing out the three equations corresponding to $i = +,-,3$, we find a set of equations

$$\begin{aligned}(\lambda+m)y^+_{j\,m+1} &= \sqrt{j(j+1)-m(m+1)}\,y^3_{jm}\,,\\ (\lambda-m)y^-_{j\,m-1} &= -\sqrt{j(j+1)-m(m-1)}\,y^3_{jm}\,,\\ (\lambda-1)y^3_{j\,m} &= \frac{1}{2}\sqrt{j(j+1)-m(m+1)}\,y^+_{j\,m+1} - \frac{1}{2}\sqrt{j(j+1)-m(m-1)}\,y^-_{jm-1}\,. \end{aligned} \qquad (5.5)$$

[9] Here, the matrix Y^i transforms in the tensor product of the spin 1 representation of SU(2) (from the explicit vector index) with the sum of the spin j representations. The eigenvectors are precisely the irreducible representations, corresponding to vector spherical harmonics.

Note that we have a separate set of equations for each value of m between $m = -j-1$ and $m = j+1$.[10] Replacing y^+ and y^- in the third equation using the first two equations, we find an equation for the eigenvalues,

$$\lambda(\lambda + j)(\lambda - (j+1)) = 0$$

so we may have $\lambda = 0$, $\lambda = -j$ or $\lambda = j+1$. Using the first two equations, we can read off the eigenstates for these three eigenvalues. For $\lambda = 0$, the eigenstates are

$$Y^i = [J^i, Y_{jm}]\,, \qquad 1 \leq j \leq N-1\,, \qquad -j \leq m \leq j$$

for a total of $N^2 - 1$ zero modes. These correspond to fluctuations in the directions of the gauge orbit and not to physical zero-modes. To see this, note that under an infinitesimal gauge transformation,

$$\frac{1}{3}J^i \longrightarrow \frac{1}{3}J^i + \frac{1}{3}i[J^i, \Lambda]$$

so that any fluctuation of the form

$$Y^i = [J^i, \Lambda]$$

corresponds to a gauge transformation. Since $[J^i, Y_{00}] = 0$, we have $N^2 - 1$ instead of N^2 gauge orbit directions, as we found above.

For the $\lambda = -j$ eigenstates, corresponding to oscillators with $M = j/3$, we find

$$\begin{aligned}
Y^+ &= -\frac{\alpha_{j-1m}}{\sqrt{N}}\sqrt{\frac{(j+m)(j+1+m)}{j(2j+1)}}Y_{j\,m+1}\,,\\
Y^- &= \frac{\alpha_{j-1m}}{\sqrt{N}}\sqrt{\frac{(j-m)(j+1-m)}{j(2j+1)}}Y_{j\,m-1}\,,\\
Y^3 &= \frac{\alpha_{j-1m}}{\sqrt{N}}\sqrt{\frac{(j+m)(j-m)}{j(2j+1)}}Y_{jm}\,,
\end{aligned} \tag{5.6}$$

where $0 < j < N$ and $-j < m < j$ and we have the reality condition $\alpha^*_{jm} = (-1)^m \alpha_{j\,-m}$.

For the $\lambda = (j+1)$ eigenstates, corresponding to oscillators with $M = \frac{j+1}{3}$, we find

$$\begin{aligned}
Y^+ &= \frac{\beta_{j+1\,m}}{\sqrt{N}}\sqrt{\frac{(j-m)(j+1-m)}{(j+1)(2j+1)}}Y_{j\,m+1}\,,\\
Y^- &= -\frac{\beta_{j+1\,m}}{\sqrt{N}}\sqrt{\frac{(j+m)(j+1+m)}{(j+1)(2j+1)}}Y_{j\,m-1}\,,\\
Y^3 &= \frac{\beta_{j+1\,m}}{\sqrt{N}}\sqrt{\frac{(j+1+m)(j+1-m)}{(j+1)(2j+1)}}Y_{j\,m}\,,
\end{aligned} \tag{5.7}$$

where $0 \leq j < N$ and $-j-1 \leq m \leq j+1$ and we have the reality condition $\beta^*_{jm} = (-1)^m \beta_{j-m}$.

[10] For $m = \pm j$, one of the equations vanishes while for $m = \pm(j+1)$, two of the equations vanish.

In terms of α and β, the quadratic lagrangian becomes

$$\begin{aligned}\mathcal{L}^Y &= \sum_{j=0}^{N-2} \frac{1}{2}|\dot{\alpha}_{jm}|^2 - \frac{1}{2}\left(\frac{j+1}{3}\right)^2 |\alpha_{jm}|^2 + \\ &+ \sum_{j=1}^{N} \frac{1}{2}|\dot{\beta}_{jm}|^2 - \frac{1}{2}\left(\frac{j}{3}\right)^2 |\beta_{jm}|^2 .\end{aligned}$$

Fermions

The quadratic fermion potential may be written as

$$V^\psi = \psi^{\dagger\alpha}\left(\frac{1}{4}\psi_\alpha + \frac{1}{3}\sigma^{i\ \beta}_{\alpha}[J^i, \psi_\beta]\right), \tag{5.8}$$

where we have suppressed SU(4) indices. Thus, we would like to find the eigenstates of the equation[11]

$$\sigma^{i\ \beta}_{\alpha}[J^i, \psi_\beta] = \lambda\psi_\beta$$

in terms of which the fermion masses will be

$$m = \left|\frac{1}{4} + \frac{1}{3}\lambda\right| .$$

Making the expansion

$$\psi_\alpha = \psi^{jm}_\alpha Y_{jm}$$

and inserting into the eigenvalue equation, we find two equations

$$\begin{aligned}(\lambda - m)\psi^{jm}_{+} &= \sqrt{(j-m)(j+1+m)}\psi^{j\ m+1}_{-} , \\ (\lambda + m + 1)\psi^{j\ m+1}_{-} &= \sqrt{(j-m)(j+1+m)}\psi^{jm}_{+} ,\end{aligned}$$

where we have labeled the SU(2) index values by $+$ and $-$ to denote the σ^3 eigenvalue. Combining these yield the equation

$$(\lambda + j + 1)(\lambda - j) = 0$$

so the masses are given by $M = \frac{1}{3}(j + \frac{3}{4})$ and $M = \frac{1}{3}(j + \frac{1}{4})$. Explicitly, the $M = \frac{1}{3}(j + \frac{3}{4})$ eigenstates are

$$\begin{aligned}\psi_+ &= \frac{\eta^{j+\frac{1}{2}\ m+\frac{1}{2}}}{\sqrt{N}}\sqrt{\frac{j+1+m}{2j+1}}Y_{jm} , \\ \psi_- &= \frac{\eta^j + \frac{1}{2}\ m + \frac{1}{2}}{\sqrt{N}}\sqrt{\frac{j-m}{2j+1}}Y_{j\ m+1} ,\end{aligned}$$

[11] In this case, we are combining the spin j representations with the spin 1/2 representation, and again, the eigenstates will simply be the irreducible representations.

Type	Label	Mass	Spins	SO(6) × SO(3) Rep	Degeneracy
$S0(6)$	x^a_{jm}	$\frac{1}{6}+\frac{j}{3}$	$0 \le j \le N-1$	$(6, 2j+1)$	$6(2j+1)$
$S0(3)$	α_{jm}	$\frac{1}{3}+\frac{j}{3}$	$0 \le j \le N-2$	$(1, 2j+1)$	$(2j+1)$
	β_{jm}	$\frac{j}{3}$	$1 \le j \le N$	$(1, 2j+1)$	$(2j+1)$
Fermions	χ^I_{jm}	$\frac{1}{4}+\frac{j}{3}$	$\frac{1}{2} \le j \le N-\frac{3}{2}$	$(\bar{4}, 2j+1)$	$4(2j+1)$
	$\eta_{I\,jm}$	$\frac{1}{12}+\frac{j}{3}$	$\frac{1}{2} \le j \le N-\frac{1}{2}$	$(4, 2j+1)$	$4(2j+1)$

Table 1: Oscillator masses for the irreducible vacuum

where we may have $0 \le j \le N-1$ and $-j-1 \le m \le j$. The $M = \frac{1}{3}(j+\frac{1}{4})$ eigenstates are

$$\psi_+ = -\frac{(\chi^\dagger)^j - \frac{1}{2}\, m + \frac{1}{2}}{\sqrt{N}} \sqrt{\frac{j-m}{2j+1}} Y_{j\ m}\,,$$
$$\psi_- = \frac{(\chi^\dagger)^j - \frac{1}{2}\, m + \frac{1}{2}}{\sqrt{N}} \sqrt{\frac{j+1+m}{2j+1}} Y_{j\ m+1}\,,$$

where $1 \le j \le N-1$ and $-j \le m \le j-1$.

In terms of the oscillators η and χ, the quadratic lagrangian for the fermions becomes

$$\mathcal{L}^\psi = \sum_{j=\frac{1}{2}}^{N-\frac{3}{2}} i\chi^\dagger_{jm} D_0 \chi_{jm} - \frac{1}{3}\left(j+\frac{3}{4}\right)\chi^\dagger_{jm}\chi_{jm} + $$
$$+ \sum_{j=\frac{1}{2}}^{N-\frac{1}{2}} i\eta^\dagger_{jm} D_0 \eta_{jm} - \frac{1}{3}\left(j+\frac{1}{4}\right)\eta^\dagger_{jm}\eta_{jm}\,.$$

Here, η and χ carry fundamental and antifundamental SU(4) indices, respectively, which we have suppressed.

Summary of oscillator masses for the irreducible vacuum

The spectrum of oscillators about the single membrane vacuum is summarized in table 1, with the masses given in units of μ. It is easy to check that the sum of the boson masses and the sum of the fermion masses are both $2/9N(8N^2+1)$, as required for the vanishing of the vacuum state zero point energy.

A nice feature of the spectrum we have calculated is that the oscillator masses are completely independent of N. As one might hope from the discretized membrane picture, N acts exactly as a short distance cutoff, providing an upper bound on the allowed values of the spin j (related to the wavelength of modes on the sphere). In particular, the spectrum has a well defined limit for large N, which should reproduce the spectrum of the quadratic part of the continuum supermembrane field theory for this background.[12]

[12]It is important to note that for any finite μ, there will be some energy above which the perturbation theory breaks down and the spectrum can no longer be trusted. We discuss this further below.

5.3 Oscillator masses in a general reducible representation

A general classical vacuum state of the theory corresponds to an N-dimensional reducible representation of SU(2), described by a block-diagonal matrix

$$X^i = \begin{pmatrix} J_1^i & & \\ & \ddots & \\ & & J_K^i \end{pmatrix},$$

where J_l^i are generators of the N_l-dimensional irreducible representation of SU(2) and $N_1 + \cdots + N_K = N$.

To deal with these states, it is convenient to break up our matrices into blocks corresponding to the individual irreducible representations, so for example

$$X_{kl}^i = \delta_{kl} J_l^i \,,$$

where $1 \le k, l \le K$ are indices labeling the blocks. Then the quadratic actions split up into a sum of terms which are of same form as for the irreducible representation above,

$$\begin{aligned} V^X &= \frac{1}{2}\left(\frac{1}{3}\right)^2 \mathrm{Tr}\left((X_{kl}^a)^\dagger \left(\frac{1}{4} X_{kl}^a + J^i \circ (J^i \circ X_{kl}^a)\right)\right), \\ V^Y &= \frac{1}{2} \left| \left(\frac{1}{3}\right)^2 (Y_{kl}^i + i\epsilon^{ijk} J^j \circ Y_{kl}^k) \right|^2, \\ V^\psi &= \psi_{kl}^{\dagger\alpha} \left(\frac{1}{4}\psi_\alpha + \frac{1}{3}\sigma^{i\ \beta}_{\alpha} J^i \circ \psi_{kl\ \beta}\right) \end{aligned}$$

but we now define

$$J^i \circ B_{kl} = J_k^i B_{kl} - B_{kl} J_l^i \,.$$

Thus, whereas the matrices in the irreducible case were in the tensor product of two N-dimensional representations of SU(2) and split up into the representations $1 \oplus 3 \oplus \cdots \oplus 2N-1$, a general rectangular block B_{kl} lies in the tensor product of the N_k and N_l-dimensional representation and may be decomposed into irreducible representations with spins $\frac{1}{2}|N_k - N_l| \le j \le \frac{1}{2}(N_k + N_l) - 1$. Thus, for example, we may write

$$X_{kl}^a = \sum (x_{kl}^a)_{jm} Y_{jm}^{N_k N_l} \,,$$

where Y's are $N_k \times N_l$ matrices which are eigenstates of J and J_3 acting as in (5.3). Using this decomposition, the diagonalization proceeds exactly as above.

The spectrum of oscillators about a general vacuum are given in table 2, again with masses in units of μ. As before, we will have zero mass oscillators from the Y^i matrices corresponding to each of the non-trivial gauge orbit directions, and these will take the form

$$Y^i = [J^i, \Lambda]$$

for arbitrary Λ. The number of non-trivial gauge orbit directions will be N^2 minus the dimension of the space of matrices that commute with J^i, or

$$N^2 - \sum M_i^2 \,,$$

where M_i is the multiplicity of the i'th irreducible representation.

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Type	Label	Mass	Spins	SO(6) × SO(3)
$S0(6)$	$(x^a_{kl})_{jm}$	$\frac{1}{6}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert \le j \le \frac{1}{2}(N_k+N_l)-1$	$(6, 2j+1)$
$S0(3)$	α^{jm}_{kl}	$\frac{1}{3}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert -1 \le j \le \frac{1}{2}(N_k+N_l)-2$	$(1, 2j+1)$
	β^{jm}_{kl}	$\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert +1 \le j \le \frac{1}{2}(N_k+N_l)$	$(1, 2j+1)$
Fermions	$\chi^{I\,jm}_{kl}$	$\frac{1}{4}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert -\frac{1}{2} \le j \le \frac{1}{2}(N_k+N_l)-\frac{3}{2}$	$(\bar{4}, 2j+1)$
	$\eta^{jm}_{I\,kl}$	$\frac{1}{12}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert +\frac{1}{2} \le j \le \frac{1}{2}(N_k+N_l)-\frac{1}{2}$	$(4, 2j+1)$

Table 2: Oscillator masses for reducible vacua

5.4 Physical states

We have now computed the spectrum of oscillators about an arbitrary vacuum of the theory. In the $\mu = \infty$ limit, one can view these classical solutions as superselection sectors, and the physical spectrum in each sector is obtained by acting on a Fock space vacuum with arbitrary numbers of creation operators for the oscillators found above. The only complication is that we must ensure that the Gauss law constraints, arising from the equation of motion for A_0, are satisfied. These N^2-1 conditions read

$$i[X^i, \Pi^i] + i[X^a, \Pi^a] + 2\psi_{I\alpha}\psi^{\dagger I\alpha} = 0\,.$$

The corresponding operator is ordered so that it becomes the generator of SU(N) transformations, and the quantum constraint requires that this operator annihilates physical states. In other words, physical states must be SU(N) invariant.

Expanding the constraint about $X^i = \mu/3RJ^i$, we find

$$\left(\frac{\mu}{3R} i[J^i, \Pi^i] + i[Y^i, \Pi^i] + i[X^a, \Pi^a] + 2\psi_{I\alpha}\psi^{\dagger I\alpha}\right)|\psi\rangle = 0\,.$$

In the limit $\mu \to \infty$, it is useful to split these N^2-1 constraints into two groups depending on whether or not the first term vanishes. Recalling that normalized oscillators in the SO(3) directions satisfy an eigenvalue equation

$$\Pi^i + i\epsilon^{ijk}[J^j, \Pi^k] = \lambda \Pi^i$$

with $M^2 = \lambda^2/3$, and commuting this equation with J^i, we find

$$\lambda[J^i, \Pi^i] = 0\,.$$

Hence, the first term in the Gauss law constraint is non-zero precisely for the momenta associated with zero-mode ($\lambda = 0$) oscillators $Y^i = [J^i, \Lambda]$ corresponding to gauge orbit directions. For these $N^2 - M_i^2$ constraints (where M_i is the multiplicity of the ith irreducible SU(2) representation), we can ignore all but the first term in the large-μ limit, and the condition is simply that the wavefunction does not depend on the coordinates corresponding to gauge orbit directions (i.e. that the zero-modes are non physical).

The remaining number of constraints, $\sum M_i^2 - 1$ is exactly the number of generators in the subgroup of SU(N) under which the vacuum state is invariant. These constraints demand that physical states are invariant under this subgroup. In a vacuum with M_i copies of the N_i-dimensional irreducible representation, the unbroken symmetry group will be $U(M_1)\times\cdots\times U(M_n)$ (including the diagonal U(1) whose Gauss law constraint is trivial). From table 2, we see that the oscillator spectrum corresponding to a given block depends only on the dimension of the two representations defining the block, so for each pair of representations N_i, N_j we will have M_iM_j identical copies of the same spectrum, which form a matrix in the bifundamental representation of $U(M_i)\times U(M_j)$ (or in the adjoint of $U(M_i)$ for the case $N_i = N_j$). The Gauss law constraint is then satisfied simply by forming $U(M_1)\times\cdots\times U(M_N)$ invariants out of the matrix creation operators. Thus, a general physical state will be obtained by acting with an arbitrary number of traces of products of these matrix creation operators on the fock space vacuum.

We have now specified the complete spectrum of physical states of the matrix model in the $\mu\to\infty$ limit.

5.5 BPS States

We have argued in section 4.1 that the unusual supersymmetry algebra of the matrix model allows the possibility of states carrying angular momentum which preserve various fractions of the 16 supersymmetries generated by Q. Now that we have the complete spectrum of the theory in the $\mu=\infty$ limit, it will be possible to explicitly identify all BPS states.

To begin, it is convenient to write the anticommutation relation for the Qs in the original SO(9) notation,

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H - \frac{\mu}{3}(\gamma^{123}\gamma^{ij})_{\alpha\beta}M^{ij} + \frac{\mu}{6}(\gamma^{123}\gamma^{ab})_{\alpha\beta}M^{ab}\,. \tag{5.9}$$

We may choose to label states by their eigenvalues for a set of Cartan generators of the SO(3)$\times$SO(6) algebra, which we take to be M^{12}, M^{45}, M^{67}, and M^{89}. The γ matrices coupling to each of these generators all commute with one another and may be represented by

$$\begin{aligned}\gamma^3 &= \sigma^3\times 1\times 1\times 1\,, &\qquad \gamma^{12345} &= 1\times\sigma^3\times 1\times 1\,,\\ \gamma^{12367} &= 1\times 1\times\sigma^3\times 1\,, &\qquad \gamma^{12389} &= -1\times\sigma^3\times\sigma^3\times 1\,,\end{aligned} \tag{5.10}$$

where the last definition is implied by

$$\gamma^{12345}\gamma^{12367}\gamma^{12389} = -\gamma^{123456789} = -1\,.$$

Any non-trivial BPS state must have non-zero angular momentum and therefore will lie in some non-trivial representation of SO(3) $\times$ SO(6). For this SO(3) $\times$ SO(6) multiplet, there will be a unique highest weight state $|\psi\rangle$, and for this state, we will have $\langle M^{AB}\rangle = 0$ for all of the non-Cartan generators of the algebra. Using the representation (5.10), the expectation value of (5.9) on this state is then

$$\langle\psi|\{Q_\alpha, Q_\beta\}|\psi\rangle = 2\delta_{\alpha\beta}\Delta_\beta\,,$$

where the sixteen diagonal elements Δ_α are given by two copies of the values

$$\Delta = H + \epsilon_1\frac{\mu}{3}M^{12} + \epsilon_2\frac{\mu}{6}M^{45} + \epsilon_3\frac{\mu}{6}M^{67} - \epsilon_2\epsilon_3\frac{\mu}{6}M^{89}$$

and $\epsilon_i = \pm 1$ may be chosen independently. Highest weight BPS states will then be exactly those for which some of these eigenvalues vanish. It is now straightforward to describe the complete set of BPS states in the spectrum by inspecting table 2.

The eigenvalues of $12(H/\mu, M^{12}/3, M^{45}/6, M^{67}/6, M^{89}/6)$ for the individual oscillators (suppressing the block indices) are

$$\begin{aligned}
\beta_{jm}:&\ (4j, 4m, 0, 0, 0)\,,\\
\alpha_{jm}:&\ (4j+4, 4m, 0, 0, 0)\,,\\
x_{jm}:&\ (4j+2, 4m, \pm 2, 0, 0),\ (4j+2, 4m, 0, \pm 2, 0),\ (4j+2, 4m, 0, 0, \pm 2)\,,\\
\chi_{jm}:&\ (4j+3, 4m, 1, 1, 1),\ (4j+3, 4m, 1, -1, -1),\\
&\ (4j+3, 4m, -1, 1, -1),\ (4j+3, 4m, -1, -1, 1)\,,\\
\eta_{jm}:&\ (4j+1, 4m, -1, -1, -1),\ (4j+1, 4m, 1, 1, -1),\\
&\ (4j+1, 4m, 1, -1, 1),\ (4j+1, 4m, -1, 1, 1)\,.
\end{aligned}$$

The $\{Q, Q\}$ eigenvalues are obtained by dotting these vectors with

$$\frac{\mu}{12}(1, \epsilon_1, \epsilon_2, \epsilon_3, -\epsilon_2\epsilon_3)\,,$$

and a state corresponding to one of these oscillators is invariant under the pair of supercharges corresponding to $(\epsilon_1, \epsilon_2, \epsilon_3)$ whenever the result is zero. It is straightforward to check that this dot product is always greater than or equal to zero (guaranteed by the BPS bound), and that it may only be zero for $m = \pm j$. For this choice, we find that each β, η, x, χ, and α oscillator commutes with 4,3,2,1, and 0 pairs of supercharges, respectively. The oscillators which preserve each supercharge are given by

$$\begin{aligned}
+++:&\ \beta_{jj},\ x_{jj}^{4-i5},\ x_{jj}^{6-i7},\ x_{jj}^{8+i9},\ \chi_{jj}^{--+},\ \eta_{jj}^{---},\ \eta_{jj}^{-++},\ \eta_{jj}^{+-+}\,,\\
++-:&\ \beta_{jj},\ x_{jj}^{4-i5},\ x_{jj}^{6+i7},\ x_{jj}^{8-i9},\ \chi_{jj}^{-+-},\ \eta_{jj}^{---},\ \eta_{jj}^{-++},\ \eta_{jj}^{++-}\,,\\
+-+:&\ \beta_{jj},\ x_{jj}^{4+i5},\ x_{jj}^{6-i7},\ x_{jj}^{8-i9},\ \chi_{jj}^{+--},\ \eta_{jj}^{---},\ \eta_{jj}^{+-+},\ \eta_{jj}^{++-}\,,\\
+--:&\ \beta_{jj},\ x_{jj}^{4+i5},\ x_{jj}^{6+i7},\ x_{jj}^{8+i9},\ \chi_{jj}^{+++},\ \eta_{jj}^{-++},\ \eta_{jj}^{+-+},\ \eta_{jj}^{++-}\,.
\end{aligned} \tag{5.11}$$

Here, we have labeled fermion oscillators by the signs of their (M^{45}, M^{67}, M^{89}) eigenvalues and denoted $x^5 + ix^6$ by x^{5+i6}. The values on the left denote the sign of ϵ_1, ϵ_2, and ϵ_3 for a given pair of supercharges. The oscillators that commute with supercharges corresponding to $\epsilon_1 = -1$ are given by an identical table with the singns on the left reversed and $m = -j$ for all oscillators instead of $m = j$.

Since all of the quantum numbers are additive, a state obtained by acting with an arbitrary combination of oscillators in a single row of the list (5.11) will be invariant under the corresponding supercharge. Thus, up to $SO(3) \times SO(6)$ rotations, all BPS states in the theory may be obtained by acting on the Fock space vacuum with an arbitrary combination of oscillators in any given row above. States for which all of the oscillators appear simultaneously in k rows of the table above will be left invariant by $2k$ supercharges. For example, up to rotations, the most general 1/4 BPS state preserving 8 supercharges will be

$$\prod_j (b^\dagger_{jj})^{n_j}|0\rangle\,,$$

where b is the creation operator associated with β. It is interesting to note that the simple 1/4 BPS states,

$$(b^\dagger_{22})^n|0\rangle$$

correspond classically to the rotating ellipsoidal membrane state found by Bak [24].

We have thus given the complete set of BPS states of the matrix model for the $\mu = \infty$ limit. It would be interesting to determine which of these remain BPS as we turn on interactions (at finite μ). This issue is discussed further in the remarks of section 8.

6. Perturbation theory

In this section, we move away from the $\mu = \infty$ limit and study perturbation theory about the free theory described in the previous section. Specifically, we would like to determine the range of parameters (N, R, μ) and energies for which perturbation theory is valid. As a first step, we rewrite the interaction hamiltonian in terms of normalized oscillators to determine the value of the "bare coupling" corresponding to different vacua. We then perform explicit calculations of the energy shift for the first non-trivial excited states about the $X = 0$ vacuum and the irreducible vacuum to determine the actual effective coupling controlling the perturbation expansion. Since these states are non-BPS, we expect their energy shifts to be generic. For fixed values of the parameters, we then estimate the height of the potential separating different vacua. This corresponds to the energy above which perturbation theory becomes inapplicable.

6.1 Expansion in terms of normalized oscillators

In the previous section, we have explicitly diagonalized the quadratic actions, writing them in terms of canonically normalized oscillators. It is useful to consider expanding the interaction hamiltonian in terms of these oscillators, in order to more accurately determine the parameter in the action suppressing interaction terms.

In general, the cubic interactions in the matrix lagrangian take the form

$$H_3 \sim \left(\frac{R}{\mu}\right)^{3/2} \mathrm{Tr}(A_1[A_2, A_3])$$

while the quartic interactions take the form

$$H_4 \sim \left(\frac{R}{\mu}\right)^{3} \mathrm{Tr}\left([A_1, A_2][A_3, A_4]\right),$$

where A_i represent any of our matrix variables.

As an example, we consider a vacuum corresponding to K copies of the N_1-dimensional irreducible representation, such that $KN_1 = N$. In this case, as discussed in section 5.3, it is convenient to split the matrices A into $N_1 \times N_1$ blocks, and we find from the analysis

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of section 5.2 that the individual blocks may be written in terms of normalized oscillators A_{jm} as

$$A^{kl} = \frac{1}{\sqrt{N_1}} c_{jm} A^{kl}_{jm} Y^{N_1}_{jm}\,, \tag{6.1}$$

where c_{jm} are coefficients independent of N. To evaluate the coefficients in the cubic and quartic actions, we note (see, for example [21, 22])

$$[Y^N_{j_1 m_1}, Y^N_{j_2 m_2}] = \frac{1}{N} b_{j_1 m_1\, j_2 m_2\, j_3 m_3} Y^N_{j_3 m_3}\,, \tag{6.2}$$

where

$$\begin{aligned} b_{j_1 m_1\, j_2 m_2\, j_3 m_3} &= (-1)^N 2N^{3/2} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\, \times \\ &\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}. \end{aligned}$$

Explicit formulae for the Wigner 3j and 6j symbols appearing here may be found in [23]. Using these, it is straightforward to show that b has a finite limit as $N \to \infty$ with corrections of order $1/N$.

Using (6.1) and (6.2) and recalling the normalization condition (5.3) for Y's, we may now rewrite the interaction hamiltonian in terms of normalized oscillators. We find schematically

$$\begin{aligned} H_3 &\sim \left(\frac{R}{\mu N_1}\right)^{3/2} bcccAAA\,, \\ H_4 &\sim \left(\frac{R}{\mu N_1}\right)^{3/2} bbccccAAAA\,. \end{aligned}$$

For large N the leading terms in the b and c coefficients are independent of N, so we conclude that the coupling corresponding to the vacuum with K spheres of radius $r = (\mu N_1/6R)$ is given by

$$g = \left(\frac{R}{\mu N_1}\right)^{3/2} \propto \frac{1}{r^{3/2}}\,.$$

For a fixed value of N, we see that the irreducible vacuum appears to have the weakest coupling, $g_{\rm irr} = (R/(\mu N))^{3/2}$ while the $X = 0$ vacuum is the most strongly coupled, with $g_{X=0} = (R/\mu)^{3/2}$.

In systems with many degrees of freedom, it is often the case that the perturbation expansion is controlled by some effective coupling which is different than the bare coupling appearing in the lagrangian. To determine the true effective coupling in our case, we now turn to some explicit perturbative calculations.

6.2 $X = 0$ vacuum

The $X = 0$ vacuum is invariant under the full U(N) symmetry group of the model. As discussed in section 5.4, it is then convenient to introduce matrix creation and annihilation

State	SO(6) $\times$ SO(3) Rep.	Energy	Degeneracy
$\|0\rangle$	$(1,1)$	0	1
$\mathrm{Tr}(A_a^\dagger)\|0\rangle$	$(6,1)$	$\frac{\mu}{6}$	6
$\mathrm{Tr}(\psi^{\dagger I\alpha})\|0\rangle$	$(\bar{4},2)$	$\frac{\mu}{4}$	8
$\mathrm{Tr}(A_i^\dagger)\|0\rangle$	$(1,3)$	$\frac{\mu}{3}$	3
$\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_b^\dagger)\|0\rangle$	$(1,1)+(20,1)$	$\frac{\mu}{3}$	$1+20$
$\mathrm{Tr}(A_a^\dagger A_b^\dagger) - \frac{1}{N}\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_b^\dagger)\|0\rangle$	$(1,1)+(20,1)$	$\frac{\mu}{3}$	$1+20$

Table 3: Lowest energy states about the $X=0$ vacuum

operators transforming in the adjoint of this symmetry group,

$$\begin{aligned} X^i &= \sqrt{\frac{3}{2}}(A_i + A_i^\dagger)\,, \\ \Pi^i &= -\frac{i}{\sqrt{6}}(A_i - A_i^\dagger)\,, \\ X^a &= \sqrt{3}(A_a + A_a^\dagger)\,, \\ \Pi^a &= -\frac{i}{\sqrt{12}}(A_a - A_a^\dagger) \end{aligned}$$

in terms of which the quadratic hamiltonian becomes

$$H_2 = \mathrm{Tr}\left(\frac{1}{3}A_i^\dagger A_i + \frac{1}{6}A_a^\dagger A_a + \frac{1}{4}\psi^{\dagger I\alpha}\psi_{I\alpha}\right).$$

Physical states satisfying the Gauss law constraint will then be obtained by acting with traces of products of these matrix creation operators on the Fock space vacuum, for example

$$\mathrm{Tr}(A_a^\dagger A_b^\dagger \psi^{\dagger I\alpha})\,\mathrm{Tr}(A_i^\dagger A_c^\dagger)|0\rangle\,.$$

The lowest lying states are given in table 3.

We can now investigate the change in energy of these states upon turning on the interaction hamiltonian, using perturbation theory in R/μ. Note that only the last state in the list above can get a shift in energy, since the other states are excitations in the U(1) part of the theory, which is free.

It is easy to see that the energy shift at first order in perturbation theory

$$\langle\psi|H_3|\psi\rangle$$

will be zero for all states, since all terms in H_3 have non-zero H_2 eigenvalue. Thus H_3 acting on any state gives a combination of basis states all of whose energies are different than that of the original state.

At second order in perturbation theory, the energy shift is given by the usual formula,

$$\begin{aligned} \Delta &= \Delta_4 + \Delta_3 \\ &= \langle\psi|H_4|\psi\rangle + \sum_n \frac{1}{E_\psi - E_n}|\langle n|H_3|\psi\rangle|^2\,. \end{aligned} \tag{6.3}$$

State	SO(6) $\times$ SO(3) Rep.	Energy	Degeneracy
$\lvert 0\rangle$	$(1,1)$	0	1
$(a^a_{00})^\dagger \lvert 0\rangle$	$(6,1)$	$\frac{\mu}{6}$	6
$\eta^{I\dagger}_{\frac{1}{2}\,m}\lvert 0\rangle$	$(\bar{4},2)$	$\frac{\mu}{4}$	8
$b^\dagger_{1m}\lvert 0\rangle$	$(1,3)$	$\frac{\mu}{3}$	3
$(a^a_{00})^\dagger (a^b_{00})^\dagger \lvert 0\rangle$	$(1,1)+(20,1)$	$\frac{\mu}{3}$	$1+20$
$a^\dagger_{00}\lvert 0\rangle$	$(1,1)$	$\frac{\mu}{3}$	1

Table 4: Lowest energy states about the irreducible vacuum

Before calculating the energy shift for the excited state, it is useful to verify explicitly that the vacuum state receives no correction. In appendix C.1, we show that for the vacuum state, $\Delta_4 = -\Delta_3 = 891/4(N^3 - N)$, so the vacuum state does remain at zero energy to second order in perturbation theory. The vacuum wavefunction does receive a correction at first order in perturbation theory, given in equation (C.3).

We now compute the vacuum shift for the lowest energy non-trivial state (i.e. involving more than U(1) oscillators),

$$|\psi\rangle = \frac{1}{\sqrt{12(N^2-1)}}\left(\mathrm{Tr}(A^\dagger_a A^\dagger_a) - \frac{1}{N}\,\mathrm{Tr}(A^\dagger_a)\,\mathrm{Tr}(A^\dagger_a)\right)|0\rangle\,.$$

This normalized state has been chosen to be orthogonal to the state

$$\mathrm{Tr}(A^\dagger_a)\,\mathrm{Tr}(A^\dagger_a))|0\rangle$$

which cannot receive any energy correction.

The calculation of the energy shift to second order in perturbation theory is performed in appendix C.1, and we find

$$\Delta = 108N\mu\left(\frac{R}{\mu}\right)^3.$$

Thus, the effective coupling controlling the perturbation expansion appears to be (ignoring numerical factors)

$$N\left(\frac{R}{\mu}\right)^3 = g^2 N\,,$$

where $g = (R/\mu)^{3/2}$ was the bare coupling suppressing the interaction terms in the action near $X = 0$.

6.3 Perturbation theory about irreducible vacuum

We next consider perturbation theory for the single membrane vacuum, corresponding to the irreducible representation of SU(2). Here, the low-energy states are obtained by acting with creation operators corresponding to the oscillators in table 1. The states with lowest energy are summarized in table 4, where we have defined a^a_{jm}, a_{jm}, and b_{jm} to be canonical annihilation operators associated with the oscillators x^a_{jm}, α_{jm}, and β_{jm}, respectively.

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Note that x^a_{00}, β_{1m} and $\eta_{\frac{1}{2}\,m}$ correspond to oscillators in the free $U(1)$ part of the theory, so it is only the last state that can receive an energy shift in perturbation theory. This is the breathing mode of the sphere.

In appendix C, we compute the leading energy shift for an arbitrary occupation number of this breathing mode, i.e., for the state

$$\frac{1}{\sqrt{n!}}(a^\dagger_{00})^n|0\rangle\,.$$

The result, coming at second order in perturbation theory from the terms (6.3) is

$$\begin{aligned}\frac{\Delta}{\mu} &= -216\binom{n}{2}\frac{R^3}{\mu^3 N(N^2-1)}\\ &= -216\binom{n}{2}g^2\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).\end{aligned}$$

(Note that for $n=1$ the energy shift Δ is zero.) In this case, we see that the leading correction for low energy states is controlled by the "bare" coupling g^2 without any additional N-dependence and is therefore finite in the large-N limit.

6.4 Validity of perturbation theory

In the previous sections, we have computed the leading correction to the energies of the first non-trivial excited states above the irreducible single-membrane vacuum and the $X=0$ vacuum. We found that the effective coupling controlling the perturbative corrections is given by $\lambda = g^2 K$, where $K=1$ and $g^2 = (\frac{R}{\mu N})^{\frac{3}{2}}$ for the single membrane case and $K=N$ and $g^2=(\frac{R}{\mu})^{\frac{3}{2}}$ for the $X=0$ vacuum.[13] Note that K corresponds to the number of coincident membranes characterizing the vacuum and in this sense λ behaves like a 't Hooft coupling.

In situations where perturbative corrections are small for low lying states, we would now like to estimate the energy above which perturbation theory breaks down. In terms of the classical potential, we expect this to occur when we move far enough away from the minimum so that the interaction terms in the potential become of the same order of magnitude as the quadratic terms.

To obtain an initial, crude estimate of this energy scale, consider starting at this irreducible vacuum and exciting the breathing mode considered in the previous section, that is, take $Y^i = \alpha\frac{\mu}{3R}J^i$. Plugging this in to the classical potential, we find

$$V=\frac{1}{2}R\,\mathrm{Tr}(J^iJ^i)\left(\frac{\mu}{3R}\right)^4\alpha^2(1+\alpha)^2=\frac{RN(N^2-1)}{8}\left(\frac{\mu}{3R}\right)^4(\alpha^2+2\alpha^3+\alpha^4)\,.$$

Thus, we see that the cubic and quartic terms in the potential will be small compared to the quadratic term when $\alpha \ll 1$. In terms of the energy, this translates to the condition

[13]The other vacua are characterized by a couplings intermediate between these values.

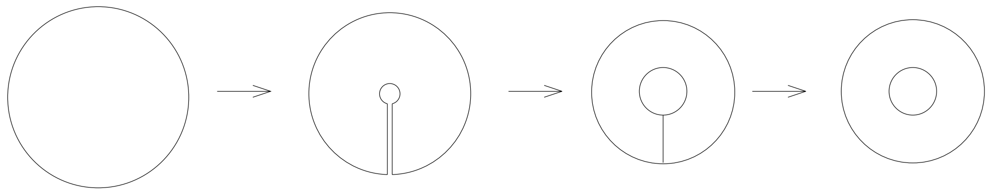

Figure 2: Low energy path between nearby vacua

(ignoring numerical factors)

$$\frac{E}{\mu} \ll \left(\frac{\mu N}{R}\right)^3 = \frac{1}{g^2}\,.$$

Our crude estimate suggests that the quadratic approximation to the potential is good for energies less than μ/g^2.

We expect this estimate to be true for a generic direction away from the minimum However, it turns out that for certain paths away from the minimum, the interaction terms can become important at a much lower energy. In appendix D, we provide an explicit path between the irreducible vacuum labeled by N and the nearby vacuum labeled by $(N-1,1)$ along which the maximum energy is of order $(\mu N/R)^3 1/N = 1/g^2 N$ rather than $1/g^2$.[14] This path corresponds to the discretized version of a continuum picture in which a thin tube of membrane extends from the south pole of the sphere to the origin where a small second sphere can grow from the tip of the tube, as depicted in figure 2.[15]

Thus, our crude estimate for the energy at which interaction terms become important was in fact too large by a factor of N. At least along a specific direction away from the minimum, we find that interactions must be important at an energy

$$\frac{E}{\mu} \sim \frac{1}{N}\left(\frac{\mu N}{R}\right)^3 = \frac{1}{g^2 N}\,.$$

From this result, one might conclude that the condition for a perturbative regime to exist should be obtained by demanding that this energy is significantly greater than the energy of the lowest lying modes, that is, $g^2 N \ll 1$. On the other hand, we have seen explicitly that the leading perturbative correction for certain low energy states was small as long as $g^2 \ll 1$, a much weaker condition. It is possible that the low energy paths between vacua we have found are very narrow for large N (a set of very small measure in the space of paths away from the minimum) and therefore that perturbation theory may be valid for larger energies than the height of these paths would indicate. In particular, from the calculation of appendix C.2, we see that the the leading perturbative correction to the breathing mode with occupation number n is small when $n^2 g^2 \ll 1$, in other words when $E \ll \mu/g$. Further work will be required to decisively determine which of these conditions (or some intermediate condition) actually governs the validity of perturbation theory.

---

[14] In terms of the natural metric $\mathrm{Tr}((X^A)^2)$ on the space of matrices, this choice for the final state represents the closest vacuum.

[15] We thank Washington Taylor for a valuable discussion about this.

7. M-theory limit

In this section, we discuss the implications of our results for the M-theory limit of the matrix model.

7.1 Implications of perturbative results

M-theory on the pp-wave background should be described using the matrix model by taking the $N \to \infty$ limit with fixed $p^+ = N/R$ and fixed μ. The resulting theory is characterized by the single boost-invariant dimensionless parameter $\mu p^+ l_p^2$. In this background, a single membrane with momentum p_i^+ in the pp-wave background will expand into a spherical BPS "giant graviton" configuration with radius μp_i^+ [6]. In general, the total momentum p^+ may be divided among any number of membranes, and each of these membranes will have a radius proportional to its momentum. This is precisely the picture we get from the large-N limit of the matrix model.

In section 5, we calculated the spectrum of the matrix model in the limit of large μ. We found that the individual oscillator masses were independent of N. For the irreducible vacuum, N acts as a cutoff controlling the maximum allowed frequency of the modes. Thus, in the large-N limit, we find a well- defined spectrum, which we may associate with the spectrum of the free part of the field theory on a single spherical supermembrane in the pp-wave background.

For reducible representations involving k copies of a single irreducible representation, we obtain k^2 copies of the spectrum for the corresponding irreducible representation, suggesting a non-abelian structure for coincident membranes. For reducible vacua involving different irreducible representations of dimension N_1 and N_2, corresponding to non-coincident membranes, we found that the spectrum corresponding to off-diagonal degrees of freedom connecting the two representations has a lower bound on oscillator masses proportional to $\mu/2|N_1 - N_2|$. Thus, if we fix the radius parameters $(\mu N_i/R)$ of the two membranes and take the large-N limit, we find that all of the off-diagonal degrees of freedom develop infinite masses, going like $N(p_1^+ - p_2^+)/p^+$.

It is not clear whether the M-theory limit is compatible with a valid perturbation expansion. In explicit perturbative calculations, we found that the leading corrections to the energies of low-lying states above the single-membrane vacuum are finite in the the M-theory limit and small when $\mu p^+ \gg 1$. On the other hand, we have shown in appendix D that there exist paths in configuration space between any two vacua with maximum energy of order $\mu(\mu p^+)^3/N$, which goes to zero in the M-theory limit. This indicates that in the large N limit, the classical vacua are no longer isolated, but rather are all connected by finite, zero-energy flat directions in the potential. Physically, these flat directions correspond to decreasing the radius (and light-cone momentum) of any membrane while increasing the radius of any other membrane at the same rate (or growing a new membrane at the origin). Quantum mechanically, these flat directions would seem to spoil any perturbative study of the model, since none of the classical vacua are long-lived and the low-energy wave-functions would presumably spread out over all the classical vacua.

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7.2 The supermembrane field theory

From the supermembrane point of view, the matrix model can be thought of as a regulated version of the light-cone supermembrane field theory on some fixed topology worldvolume. When interpreted in this context, our results have a number of interesting implications.

To be definite, consider starting from the supermembrane hamiltonian (2.27), and choose a spherical worldvolume of radius 1 labeled by coordinates x_1, x_2, x_3 with $x_1^2+x_2^2+x_3^2=1$.[16] The potential term in the bosonic part of the hamiltonian is

$$V \propto \left(\epsilon^{ijk}\{X^i,X^j\} - \frac{2\mu p^+}{3}X^k\right)^2 + \{X^a,X^b\}^2 + 2\{X^i,X^a\}^2 + \left(\frac{\mu p^+}{3}\right)^2 (X^a)^2\,, \qquad (7.1)$$

where $\{A,B\} = \epsilon^{ijk}x_i\partial_j A\partial_k B$ is the canonical Poisson bracket. This potential has a minimum when $X^a=0$ and

$$\{X^i,X^j\} = \epsilon^{ijk}\frac{\mu p^+}{3}X^k\,. \qquad (7.2)$$

In the continuum case, there are only two vacua corresponding to non-singular field configurations, namely $X=0$ and $X^i = R_{\rm sph}x^i$ where $R_{\rm sph} = \mu p^+/6$.

By analogy with the matrix model analysis above, one might have been interested to study the field theory expanded about the large membrane solution. The resulting theory has an unusual gauge symmetry coming from the symmetry under area-preserving diffeomorphisms,

$$\delta Y^i = -R_{\rm sph}\epsilon^{ijk}x^j\partial^k\Lambda + \{Y^i,\Lambda\}\,.$$

One may attempt to decompose Y into fields with more standard gauge transformation properties as

$$\phi = \frac{1}{R_{\rm sph}}\,x\cdot\vec{Y}\,, \qquad \vec{A} = \frac{1}{R_{\rm sph}}\,x\times\vec{Y}\,. \qquad (7.3)$$

To leading order, the fields A and ϕ have the usual transformation properties of a gauge field and a scalar, and furthermore A appears in the action as a massless gauge field while ϕ is massive. However, there are also non-linear terms in the gauge transformation which may only be removed by adding higher order terms in the definition of A and ϕ. These introduce even higher order terms in the gauge transformation, and so forth, indicating that the definition of a proper gauge field with transformation property $\delta A_i = \partial_i\Lambda - \{A_i,\Lambda\}$ would require an infinite series of terms in Y. Thus, for now we shall consider the theory written in terms of Y.

At first sight, this theory looks non-renormalizable, since the $\{\ ,\ \}^2$ terms have dimension 6, suggesting for example that one would find divergent one-loop contributions in the quadratic effective action going like Λ^5. In the matrix regularized theory, the role of the UV cutoff is played by N, so we might have expected N^5 behavior in the mass shift we have calculated for the breathing mode. In reality, our result is $\mathcal{O}(1)$ in the large-N limit, indicating that the theory may have much better renormalizability properties than it appears to. This is at least partly due to supersymmetric cancellations. Another possible

[16] In terms of the usual spherical coordinates ϕ,θ, we have $x_3=\cos\theta$, $x_1=\sin\theta\cos\phi$, and $x_2=\sin\theta\sin\phi$ and the Poisson bracket on the sphere is defined by $\{\theta,\phi\}=1/\sin\theta$.

explanation is that we should really consider the theory as a $\theta \to 0$ limit of a noncommutative 2+1 dimensional field theory on a sphere with only quartic interactions. Such a theory would be expected to give only linear divergences in the quadratic effective action, or even weaker behaviour in the supersymmetric case. It would be quite interesting if the Poisson bracket field theory turned out to be perturbatively renormalizable.

We noted that the supermembrane field theory has two non-singular classical vacuum solutions, however, from the matrix theory point of view, we have seen that the theory has a continuum of other vacua connected to these by finite zero-energy flat directions. In the continuum theory, these vacua will appear as singular field configurations. To understand what these look like, recall that one path between the single membrane vacuum and a vacuum with two spheres involved forming a thin tube of membrane from the south pole of the original sphere which moves up to the origin and nucleates another sphere. From the membrane field theory point of view, this corresponds to nucleating a small bubble of domain wall at some point on the sphere, where the inside of the domain wall maps to the new sphere, and the domain wall itself maps to the tube. In the final vacuum state with two spheres, the tube has shrunk to zero size, so the domain wall becomes singular. Thus, the flat directions we have found correspond to the existence of massless singular domain walls, or tensionless strings, in the supermembrane field theory. Presumably, these non-perturbative massless excitations make the theory extremely difficult to study.

8. Summary and Discussion

In this paper, we have studied the matrix model proposed by Berenstein, Maldacena, and Nastase to describe M-theory on the maximally supersymmetric pp-wave solution of eleven-dimensional supergravity. We have seen that the model may be derived directly as a regularized form of the light-cone supermembrane action in the pp-wave background, or alternatively (with certain assumptions) from D0-branes in a corresponding type-IIA background.

We noted that for large values of the mass parameter μ, the theory may be expanded about its classical vacua to give a quadratic theory with interactions suppressed by powers of $1/\mu$. In the large-μ limit for fixed N and R, we determined the complete physical spectrum of the theory and found that the individual energy levels are rational multiples of μ independent of N. We characterized the complete set of BPS states in this limit, finding infinite towers of states preserving 2,4,6, and 8 of the 32 supercharges (in addition to the 1/2 BPS vacuum states). We then considered perturbation theory about the vacua, and found that vacua corresponding to more spheres at smaller radii are generally more strongly coupled. For the most weakly coupled vacuum, corresponding to a single membrane sphere, we found that the leading corrections to the energies of low-lying excited states were small when $g^2 = 1/(\mu p^+)^3 \ll 1$, independent of N. On the other hand, we saw that the various vacua are connected by paths along which the maximum energy is $(\mu p^+)^3/N$ indicating a more restricted regime of validity for perturbation theory incompatible with the M-theory limit.

A very interesting direction to pursue would be to establish which of the BPS states appearing in the $\mu = \infty$ limit remain supersymmetric for finite μ. Typically, BPS multiplets may only become non BPS by combining with other BPS multiplets to form larger non-supersymmetric representations. By better understanding the representation theory of the pp-wave superalgebra, it may be possible to show that certain BPS multiplets are protected. The existence of protected states would provide reliable information about the spectrum even in the regime where perturbation theory is invalid. Another approach to determining which states remain BPS would be to perform a direct calculation of the energy shift in perturbation theory, as we have done in this paper for two simple non-BPS states. Since the energy of BPS states is determined exactly by their angular momenta, any state which remains BPS when interactions are turned on must have zero energy shift in perturbation theory. We note that at least one of the 1/4 BPS states we have found at $\mu = \infty$ should remain BPS, namely the rotating ellipsoidal membrane solution considered in [24].

We would now like to briefly address the issue of vacuum stability. We have seen that classically, and to leading orders in perturbation theory, the various vacua are all degenerate and have zero energy. It is not clear whether this will be true nonperturbatively in the full quantum theory. We know that for $\mu = 0$ there exists precisely one normalizable zero-energy state, however it is possible that other zero-energy states (corresponding to multiple gravitons) exist which become non normalizable as $\mu \to 0$. If the classical vacua do not remain degenerate, then the exact wavefunctions of the theory will presumably not be localized around any one of the classical vacua, and transitions will occur. On the other hand, if μ is large, the low-energy wavefunctions we have been discussing should serve as useful approximate wavefunctions over time scales much shorter than the typical decay rate from one vacuum to another. We expect that for large values of μ, the path integral governing the transition rate between different minima of the classical potential (corresponding to membrane transition amplitudes) will be dominated by classical instanton solutions so that the rates may be estimated by a stationary phase approximation. Some examples of these instanton solutions have been worked out in [25].

It is interesting to note that the more commonly studied case of type-IIB string theory on the maximally supersymmetric pp-wave solution of type-IIB supergravity should behave very much like the M-theory case when the string coupling is taken to be of order 1. In particular, along the range of $\mu p^+ \alpha'$ with fixed $g_s = 1$, the classical vacua of the theory are characterized by collections of D3-brane giant gravitons with the sum of the squared radii proportional to μp^+ (in units where $\alpha' = 1$). It would be interesting to determine whether a matrix model for the DLCQ of type-IIB string theory on this background (discussed recently in [26]) also contains a perturbative regime for large μ.

Acknowledgments

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A. Fermions in $\mathrm{SU}(2) \times \mathrm{SU}(4)$ notation.

The massive matrix model preserves an $\mathrm{SO}(3) \times \mathrm{SO}(6) \sim \mathrm{SU}(2) \times \mathrm{SU}(4)$ subgroup of the original $\mathrm{SO}(9)$ symmetry of the BFSS model. It is convenient to write the action in a form for which the $\mathrm{SO}(3) \times \mathrm{SO}(6)$ transformation properties of all fields are manifest. This is already the case for the bosons, so we turn to the fermionic fields. Under the decomposition

$$\mathrm{SO}(9) \longrightarrow \mathrm{SO}(6) \times \mathrm{SO}(3)$$

the spinor representation splits up as

$$\begin{aligned} \mathbf{16} &\longrightarrow (\mathbf{4}, \mathbf{2}) + (\bar{\mathbf{4}}, \bar{\mathbf{2}}) \,, \\ \Psi &\longrightarrow \psi_{I\alpha}, \tilde{\psi}^{J\beta} \,, \end{aligned}$$

where we take I as a fundamental $\mathrm{SU}(4)$ index and α as a fundamental $\mathrm{SU}(2)$ index. Our spinors obey a reality condition, which in the reduced notation becomes simply

$$(\psi^\dagger)^{I\alpha} = \tilde{\psi}^{I\alpha} \,.$$

To decompose the terms in the action, we introduce the matrices g^a_{IJ} which relate the antisymmetric product of two $\mathbf{4}$ representations of $\mathrm{SU}(4)$ to the vector of $\mathrm{SO}(6)$. Then, we may choose[17]

$$\Psi = \begin{pmatrix} \psi_{I\alpha} \\ \epsilon_{\alpha\beta}\psi^{\dagger I\beta} \end{pmatrix}$$

and

$$\gamma^a = \begin{pmatrix} 0 & 1 \times \mathrm{g}^a \\ 1 \times (\mathrm{g}^a)^\dagger & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} -\sigma^i \times 1 & 0 \\ 0 & \sigma^i \times 1 \end{pmatrix}.$$

These Dirac matrices satisfy the Clifford algebra as long as we take normalizations so that

$$\mathrm{g}^a(\mathrm{g}^b)^\dagger + \mathrm{g}^b(\mathrm{g}^a)^\dagger = 2\delta^{ab} \,.$$

We may then rewrite the quadratic fermion terms in our original action as[18]

$$\begin{aligned} \frac{i}{2}\Psi^\dagger D_0 \Psi &\longrightarrow i\psi^{\dagger I\alpha} D_0 \psi_{I\alpha} \,, \\ -\frac{i}{8}\Psi^\dagger \gamma^{123}\Psi &\longrightarrow -\frac{1}{4}\psi^{\dagger I\alpha}\psi_{I\alpha} \,, \\ \frac{1}{2}\Psi^\dagger \gamma^i [X^i, \Psi] &\longrightarrow -\psi^{\dagger I\alpha}\sigma^{i\ \beta}_{\alpha}[X^i, \psi_{I\beta}] \,, \\ \frac{1}{2}\Psi^\dagger \gamma^a [X^a, \Psi] &\longrightarrow \frac{1}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha I}\mathrm{g}^a_{IJ}[X^a, \psi^{\dagger\beta J}] - \frac{1}{2}\epsilon^{\alpha\beta}\psi_{\alpha I}(\mathrm{g}^\dagger)^{aIJ}[X^a, \psi_{\alpha J}] \,. \end{aligned}$$

[17] In this representation, the Dirac matrices and fermions are not real, however the spinors still obey a reality condition which reduces them to 16 independent real components.

[18] Since we are working with fermions that are not explicitly real, we need to replace $\Psi^\top$ with $\Psi^\dagger$.

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B. Symmetry algebra in terms of matrix generators.

In this appendix, we review the symmetry algebra of the matrix model and provide explicit expressions for the generators in terms of the matrix model variables.

The bosonic generators include the hamiltonian H, the light-cone momentum P^+ (a central term in the algebra) and rotation generators of the $\mathrm{SO}(3)\times\mathrm{SO}(6)$ symmetry which we denote by M^{ij} and M^{ab}, respectively. In addition, there are generators a_a and a_i with non-vanishing commutation relations

$$\begin{aligned}
\left[a_a, a_b^\dagger\right] &= P^+\delta_{ab}\,,\\
\left[a_i, a_j^\dagger\right] &= P^+\delta_{ij}\,,\\
[H, a_i] &= -\frac{\mu}{3}a_i\,,\\
[H, a_a] &= -\frac{\mu}{6}a_a\,,\\
\left[M^{ab}, a_c\right] &= i\delta_{ac}a_b - i\delta_{bc}a_a\,,\\
\left[M^{ij}, a_k\right] &= i\delta_{ik}a_j - i\delta_{jk}a_i\,.
\end{aligned}$$

These are the creation and annihilation operators corresponding to the decoupled U(1) part of the theory which describes a particle (the center of mass coordinate) in a harmonic potential.

The algebra has in addition 32 fermionic generators. The first 16, labeled by $q_{I\alpha}$ change the overall polarization state in the U(1) part of the theory, while the other 16, labeled by $Q_{I\alpha}$ act non-trivially.

These generators obey anticommutation relations

$$\begin{aligned}
\{Q^{\dagger I\alpha}, Q_{J\beta}\} &= 2\delta^I_J\delta^\alpha_\beta H - \frac{\mu}{3}\epsilon^{ijk}\sigma^{k\alpha}_\beta\delta^I_J M^{ij} - \frac{i\mu}{6}\delta^\alpha_\beta(\mathrm{g}^{ab})_J{}^I M^{ab}\,,\\
\{q_{I\alpha}, Q_{J\beta}\} &= -i\sqrt{\frac{\mu}{3}}\mathrm{g}^a_{IJ}\epsilon_{\alpha\beta}a_a\,,\\
\{q^{\dagger I\alpha}, Q_{J\beta}\} &= -i\sqrt{\frac{2\mu}{3}}\sigma^{i\,\alpha}_\beta\delta^I_J a_i^\dagger\,,\\
\{q^{\dagger I\alpha}, q_{J\beta}\} &= \delta^\alpha_\beta\delta^I_J P^+
\end{aligned}$$

and have non-vanishing commutation relations with the bosonic generators given by

$$\begin{aligned}
[H, q_{I\alpha}] &= -\frac{\mu}{4}q_{I\alpha}\,,\\
[H, Q_{I\alpha}] &= \frac{\mu}{12}Q_{I\alpha}\,,\\
\left[Q_{I\alpha}, a_a^\dagger\right] &= -i\sqrt{\frac{\mu}{3}}\mathrm{g}^a_{IJ}\epsilon_{\alpha\beta}q^{\dagger J\beta}\,,\\
[Q_{I\alpha}, a_i] &= \sqrt{\frac{2\mu}{3}}i\sigma^{i\,\beta}_\alpha q_{I\beta}
\end{aligned}$$

with additional commutation relations between the fermionic generators and M's appropriate to the $\mathrm{SO}(3)\times\mathrm{SO}(6)$ transformation properties of the qs and Qs.

These generators are realized explicitly in the matrix model as

$$
\begin{aligned}
P^+ &= \frac{1}{R}\,\mathrm{Tr}(1)\,,\\
a_i &= \frac{1}{\sqrt{R}}\,\mathrm{Tr}\left(\sqrt{\frac{\mu}{6R}}X^i + \sqrt{\frac{3R}{2\mu}}i\Pi^i\right),\\
a_a &= \frac{1}{\sqrt{R}}\,\mathrm{Tr}\left(\sqrt{\frac{\mu}{12R}}X^a + \sqrt{\frac{3R}{\mu}}i\Pi^a\right),\\
M^{ij} &= \mathrm{Tr}(X^i\Pi^j - X^j\Pi^i + i\epsilon^{ijk}\psi^\dagger\sigma^k\psi)\,,\\
M^{ab} &= \mathrm{Tr}(X^a\Pi^b - X^b\Pi^a + \frac{1}{2}\psi^\dagger \mathsf{g}^{ab}\psi)\,,\\
q_{I\alpha} &= \frac{1}{\sqrt{R}}\,\mathrm{Tr}(\psi_{I\alpha})\,,\\
Q_{I\alpha} &= \sqrt{R}\,\mathrm{Tr}\left(\left(\Pi^a - i\frac{\mu}{6R}X^a\right)\mathsf{g}^a_{IJ}\epsilon_{\alpha\beta}\psi^{\dagger J\beta} - \left(\Pi^i + i\frac{\mu}{3R}X^i\right)\sigma^{i\ \beta}_{\alpha}\psi_{I\beta}+\right.\\
&\quad\left.+\frac{1}{2}[X^i,X^j]\epsilon^{ijk}\sigma^{k\beta}_{\alpha}\psi_{I\beta} - \frac{i}{2}[X^a,X^b](\mathsf{g}^{ab})_I^{\,J}\psi_{J\alpha} + i[X^i,X^a]\sigma^i\mathsf{g}^a_{IJ}\epsilon_{\alpha\beta}\psi^{\dagger J\beta}\right),
\end{aligned}
$$

where $\mathsf{g}^{ab} = \frac{1}{2}(\mathsf{g}^a\mathsf{g}^{\dagger b} - \mathsf{g}^b\mathsf{g}^{\dagger a})$. The hamiltonian H may be read off from the action in section 4.1. Using the quantum commutation relations

$$
[X^i_{kl},\Pi^j_{mn}] = i\delta^{ij}\delta_{kn}\delta_{lm}\,,\qquad [X^a_{kl},\Pi^b_{mn}] = i\delta^{ab}\delta_{kn}\delta_{lm}\,,\qquad \{\psi^{\dagger I\alpha}_{kl},(\psi_{J\beta})_{mn}\} = \delta^I_J\delta^\alpha_\beta\delta_{kn}\delta_{lm}
$$

we may verify explicitly that the algebra given above is satisfied.

For convenience, we also provide the supersymmetry algebra in the original SO(9) notation of BMN,

$$
\begin{aligned}
\{Q_\alpha,Q_\beta\} &= 2\delta_{\alpha\beta}H - \frac{\mu}{3}(\gamma^{123}\gamma^{ij})_{\alpha\beta}M^{ij} + \frac{\mu}{6}(\gamma^{123}\gamma^{ab})_{\alpha\beta}M^{ab}\,,\\
\{Q_\alpha,q_\beta\} &= i\sqrt{\frac{\mu}{3}}\left(\left\{\frac{1}{2}(1-i\gamma^{123})\gamma^a\right\}_{\alpha\beta}a^\dagger_a - \left\{\frac{1}{2}(1+i\gamma^{123})\gamma^a\right\}_{\alpha\beta}a_a\right) -\\
&\quad - i\sqrt{\frac{2\mu}{3}}\left(\left\{\frac{1}{2}(1-i\gamma^{123})\gamma^i\right\}_{\alpha\beta}a_i - \left\{\frac{1}{2}(1+i\gamma^{123})\gamma^i\right\}_{\alpha\beta}a^\dagger_i\right),\\
\{q_\alpha,q_\beta\} &= \delta_{\alpha\beta}P^+\,,
\end{aligned}
$$

where

$$
\begin{aligned}
q &= \frac{1}{\sqrt{R}}\,\mathrm{Tr}(\Psi)\,,\\
Q &= \sqrt{R}\,\mathrm{Tr}\left(\Pi^a\gamma^a\Psi + \Pi^i\gamma^i\Psi - \frac{i}{2}[X^i,X^j]\gamma^{ij}\Psi - \frac{i}{2}[X^a,X^b]\gamma^{ab}\Psi - i[X^i,X^a]\gamma^{ia}\Psi-\right.\\
&\quad\left.- \frac{\mu}{3R}X^i\gamma^{123}\gamma^i\Psi + \frac{\mu}{6R}X^a\gamma^{123}\gamma^a\Psi\right).
\end{aligned}
$$

C. Perturbative calculations

In this appendix, we perform explicitly the calculation of the energy shift for the lowest non-trivial excited states above the $X = 0$ vacuum and the irreducible vacuum.

C.1 $X = 0$ vacuum

Writing out the cubic interaction in terms of the matrix creation and annihilation operators, we find

$$\begin{aligned} H_3 = \left(\frac{R}{\mu}\right)^{3/2} \mathrm{Tr} \Bigg(& \sqrt{\frac{3}{8}} i\epsilon^{ijk} (A_i A_j A_k + 3A_i^\dagger A_j A_k + 3A_i^\dagger A_j^\dagger A_k + A_i^\dagger A_j^\dagger A_k^\dagger) - \\ & - \sqrt{\frac{3}{2}} \psi^{\dagger\alpha} \sigma^{i\ \beta}_{\alpha} [A_i + A_i^\dagger, \psi_\beta] - \frac{\sqrt{3}}{2} \epsilon_{\alpha\beta} \psi^{\dagger\alpha} \mathrm{g}^a [A_a + A_a^\dagger, \psi^{\dagger\beta}] + \\ & + \frac{\sqrt{3}}{2} \epsilon^{\alpha\beta} \psi_\alpha \mathrm{g}^{\dagger a} [A_a + A_a^\dagger, \psi_\beta] \Bigg) . \end{aligned}$$

For the quartic term, it will be more convenient to keep the coordinate notation

$$H_4 = -\left(\frac{R}{\mu}\right)^3 \mathrm{Tr} \left(\frac{1}{4} [X^i, X^j]^2 + \frac{1}{2} [X^i, X^a]^2 + \frac{1}{4} [X^a, X^b]^2 \right) .$$

It is easy to see that the energy shift at first order in perturbation theory

$$\langle \psi | H_3 | \psi \rangle$$

will be zero for all states, since all terms in H_3 have non-zero H_2 eigenvalue. Thus H_3 acting on any state gives a combination of basis state all of whose energies are different than that of the original state.

At second order in perturbation theory, the energy shift is given by the usual formula,

$$\begin{aligned} \Delta &= \Delta_4 + \Delta_3 \\ &= \langle \psi | H_4 | \psi \rangle + \sum_n \frac{1}{E_\psi - E_n} |\langle n | H_3 | \psi \rangle|^2 . \end{aligned} \tag{C.1}$$

We start by verifying that the vacuum state stays at zero energy. The contribution to the vacuum shift from H_4 is given by

$$\begin{aligned} \Delta_4 &= \langle 0 | H_4 | 0 \rangle \\ &= \frac{27}{4} (N^3 - N) + 135 (N^3 - N) + 81 (N^3 - N) , \end{aligned}$$

where the three terms come from the ij, ab, and ai terms in the quartic potential, respectively.

To compute Δ_3, note that

$$H_3 | 0 \rangle = \sqrt{\frac{3}{8}} i \sqrt{18(N^3 - N)} | A \rangle + \sqrt{3} \sqrt{48(N^3 - N)} | B \rangle , \tag{C.2}$$

where we have defined orthonormal states

$$\begin{aligned} | A \rangle &= \frac{1}{18(N^3 - N)} \epsilon^{ijk} \, \mathrm{Tr}(A_i^\dagger A_j^\dagger A_k^\dagger) | 0 \rangle , \\ | B \rangle &= \frac{1}{48(N^3 - N)} \epsilon_{\alpha\beta} \mathrm{g}^a_{IJ} \, \mathrm{Tr}(\psi^{\dagger I\alpha} \psi^{\dagger J\beta} A_a^\dagger) | 0 \rangle . \end{aligned}$$

Thus, the contribution to the vacuum shift from H_3 is

$$\begin{aligned}\Delta_3 &= -\frac{1}{E_A}|\langle A|H_3|0\rangle|^2 - \frac{1}{E_B}|\langle B|H_3|0\rangle|^2 \\ &= -\frac{27}{4}(N^3-N) - 216(N^3-N) \\ &= -\Delta_4\,.\end{aligned}$$

Thus, we have shown that the vacuum state remains at zero energy to second order in perturbation theory. Using (C.2), we may also compute the correction to the vacuum wavefunction to first order in perturbation theory, and we find the result

$$\begin{aligned}|\mathrm{vac}\rangle &= \frac{1}{\sqrt{1+\frac{891}{4}\lambda^2(N^3-N)}}\times \\ &\times\left(|0\rangle - \sqrt{\frac{3}{8}}i\lambda\sqrt{18(N^3-N)}|A\rangle - 18\lambda\sqrt{(N^3-N)}|B\rangle\right). \end{aligned} \tag{C.3}$$

We now compute the vacuum shift for the lowest energy non-trivial state (i.e. involving more than U(1) oscillators),

$$|\psi\rangle = \frac{1}{\sqrt{12(N^2-1)}}\left(\mathrm{Tr}(A_a^\dagger A_a^\dagger) - \frac{1}{N}\,\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_a^\dagger)\right)|0\rangle\,.$$

This normalized state has been chosen to be orthogonal to the state

$$\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_a^\dagger))|0\rangle$$

which cannot receive any energy correction.

To compute Δ_3, we note that H_3 may be written as a sum of terms H_3^i with specific H_2 eigenvalues,

$$H_3 = \sum H_3^i$$

such that

$$[H_2, H_3^i] = E_i H_3^i\,,$$

where H_2 is the quadratic hamiltonian and all the E_i's are distinct.

Using this decomposition, we find

$$\begin{aligned}\Delta_3 &= \sum_n \frac{1}{E_\psi - E_n}|\langle n|H_3|\psi\rangle|^2 \\ &= \sum_{n,i,j} -\frac{1}{E_i}\langle\psi|(H_3^j)^\dagger|n\rangle\langle n|H_3^i|\psi\rangle \\ &= \sum_{n,i} -\frac{1}{E_i}\langle\psi|(H_3^i)^\dagger|n\rangle\langle n|H_3^i|\psi\rangle \\ &= \sum_i -\frac{1}{E_i}\langle\psi|(H_3^i)^\dagger H_3^i|\psi\rangle\,.\end{aligned}$$

Thus, we may write Δ as

$$\Delta = \langle\psi|H_4|\psi\rangle - \sum_i \frac{1}{E_i}\langle\psi|(H_3^i)^\dagger H_3^i|\psi\rangle\,. \tag{C.4}$$

Since the vacuum energy shift was zero, we can ignore all "disconnected" contributions in which the creation operators from the initial state contract only with annihilation operators from the final state since these terms will be the same as for the vacuum shift. A further simplification arises from the fact that the interaction vertices are written in terms of commutators of X's with the result that $\mathrm{Tr}(A^\dagger)$ or $\mathrm{Tr}(A)$ from the initial or final state contracted with any of the interaction terms will vanish. Thus, we may ignore the

$$-\frac{1}{N}\,\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_a^\dagger)$$

piece since it contributes only to the disconnected contributions.

The connected contributions to Δ_3 come from the terms

$$\begin{aligned} H_3^1 &= \sqrt{3}\epsilon_{\alpha\beta}\mathsf{g}^a_{IJ}\,\mathrm{Tr}(\psi^{\dagger\alpha I}\psi^{\dagger\beta J}A_a)\,, \\ H_3^2 &= \sqrt{3}\epsilon_{\alpha\beta}\mathsf{g}^a_{IJ}\,\mathrm{Tr}(\psi^{\dagger\alpha I}\psi^{\dagger\beta J}A_a^\dagger) \end{aligned}$$

with H_2 eigenvalues $E_1 = 1/3$, and $E_2 = 2/3$. The two contributions to Δ_3 are then

$$\begin{aligned} \Delta_3^1 &= -3\langle\psi|(H_3^1)^\dagger H_3^1|\psi\rangle \\ &= -3\cdot\frac{1}{12(N^2-1)}\cdot 3|2\epsilon_{\alpha\beta}\mathsf{g}^a_{IJ}\,\mathrm{Tr}(\psi^{\dagger\alpha I}\psi^{\dagger\beta J}A_a)|0\rangle|^2 \\ &= -144N \end{aligned}$$

and

$$\begin{aligned} \Delta_3^2 &= -\frac{3}{2}\langle\psi|(H_3^2)^\dagger H_3^2|\psi\rangle \\ &= -\frac{3}{2}\cdot\frac{1}{12(N^2-1)}\cdot 3|\epsilon_{\alpha\beta}\mathsf{g}^a_{IJ}\,\mathrm{Tr}(\psi^{\dagger\alpha I}\psi^{\dagger\beta J}A_a^\dagger)\,\mathrm{Tr}(A_b^\dagger A_b^\dagger)|^2 \\ &= -72N\,. \end{aligned}$$

The connected part of Δ_4 also receives two contributions, coming from

$$H_4^1 = -\frac{1}{2}\,\mathrm{Tr}(X^aX^bX^aX^b - X^aX^aX^bX^b)$$

and

$$H_4^2 = -\,\mathrm{Tr}(X^aX^iX^aX^i - X^aX^aX^iX^i)\,.$$

The computation may be carried out efficiently using diagrams with double-line notation, making use of the propagators

$$\begin{aligned} \langle X^i_{kl}X^j_{mn}\rangle &= \frac{3}{2}\delta^{ij}\delta_{kn}\delta_{lm}\,, \\ \langle X^a_{kl}X^b_{mn}\rangle &= 3\delta^{ab}\delta_{kn}\delta_{lm}\,, \\ \langle X^i_{kl}A^j_{mn}\rangle &= \langle (A^\dagger)^i_{kl}X^j_{mn}\rangle = \sqrt{\frac{3}{2}}\delta^{ij}\delta_{kn}\delta_{lm}\,, \\ \langle X^a_{kl}A^b_{mn}\rangle &= \langle (A^\dagger)^a_{kl}X^b_{mn}\rangle = \sqrt{3}\delta^{ab}\delta_{kn}\delta_{lm}\,. \end{aligned}$$

Evaluating the various figure eight diagrams we find

$$\begin{aligned}\Delta_4^1 &= -\frac{1}{12(N^2-1)}\cdot\frac{1}{2}\langle \mathrm{Tr}(A_cA_c)\,\mathrm{Tr}(X^aX^bX^aX^b - X^aX^aX^bX^b)\,\mathrm{Tr}(A_d^\dagger A_d^\dagger)\rangle \\ &= 270N\end{aligned}$$

and

$$\begin{aligned}\Delta_4^2 &= -\frac{1}{12(N^2-1)}\langle \mathrm{Tr}(A_cA_c)\,\mathrm{Tr}(X^aX^iX^aX^i - X^aX^aX^iX^i)\,\mathrm{Tr}(A_d^\dagger A_d^\dagger)\rangle \\ &= 54N\,.\end{aligned}$$

Thus, we find that the total energy shift for the lowest non-trivial excited state $|\psi\rangle$ is given by

$$\begin{aligned}\Delta &= \Delta_3^1+\Delta_3^2+\Delta_4^1+\Delta_4^2 \\ &= 108N\mu\left(\frac{R}{\mu}\right)^3 .\end{aligned}$$

Thus, the effective coupling controlling the perturbation expansion appears to be (ignoring numerical factors)

$$\lambda_{X=0} = N\left(\frac{R}{\mu}\right)^3 = g^2N\,,$$

where $g=(R/\mu)^{3/2}$ was the bare coupling suppressing the interaction terms in the action near $X=0$.

C.2 Irreducible vacuum

We now compute the energy shift for the an arbitrary occupation number of the breathing mode of the membrane sphere for irreducible vacuum,

$$|\psi\rangle = \frac{1}{\sqrt{n!}}(a_{00}^\dagger)^n|0\rangle\,,$$

where $a^\dagger$ is the creation operator associated with the breathing mode α_{00} of the sphere.

We want to calculate the energy shift upon turning on the interaction terms

$$\begin{aligned}H_3 &= \left(\frac{R}{\mu}\right)^{3/2}\mathrm{Tr}\left(-\frac{1}{3}[J^i,X^a][Y^i,X^a]-\frac{1}{3}[J^i,Y^j][Y^i,Y^j]+\frac{i}{3}\epsilon^{ijk}Y^iY^jY^k+\right.\\ &\qquad\left.+\psi^\dagger\sigma^i[Y^i,\psi]-\frac{1}{2}\epsilon\psi^\dagger\mathsf{g}^a[X^a,\psi^\dagger]+\frac{1}{2}\epsilon\psi(\mathsf{g}^\dagger)^a[X^a,\psi]\right)\,,\\ H_4 &= \left(\frac{R}{\mu}\right)^3\mathrm{Tr}\left(-\frac{1}{4}[Y^i,Y^j]^2-\frac{1}{4}[X^a,X^b]^2-\frac{1}{2}[Y^i,X^a]^2\right).\end{aligned}\tag{C.5}$$

As in our calculation for the irreducible vacuum, the leading correction to the energy will come at second order in perturbation theory and will be given by

$$\Delta = \langle\psi|H_4|\psi\rangle - \sum_i\frac{1}{E_i}\langle\psi|(H_3^i)^\dagger H_3^i|\psi\rangle\,,\tag{C.6}$$

where, as discussed in the previous section, we have expanded

$$H_3 = \sum H_3^i$$

such that

$$[H_2, H_3^i] = E_i H_3^i\,,$$

where H_2 is the quadratic hamiltonian and all the E_i's are distinct.

Again, we need only evaluate the connected contributions, since the disconnected pieces will be the same as for the vacuum state and therefore cancel. For all connected contributions, the as and $a^\dagger$s from the initial and final state must contract with some Ys in H_3 or H_4, and these Wick contractions may be performed using the relations

$$\langle a Y^i_{mn}\rangle = \langle Y^i_{mn} a^\dagger\rangle = \sqrt{\frac{6}{N(N^2-1)}} J^i_{mn}\,.$$

In other words, Wick contracting a Y with a or $a^\dagger$ replaces it with a J.

The connected contributions may be divided as

$$\Delta = \tilde{\Delta} + \bar{\Delta}$$

where $\tilde{\Delta}$ denotes the contributions for which all but one of the as from the final state contract with $a^\dagger$s from the initial state, while $\bar{\Delta}$ denotes those for which all but two of the as from the final state contract with $a^\dagger$s from the initial state. Taking into account the normalization factor for the state and the combinatorical factors associated with pairing as from the final state with $a^\dagger$s from the initial state, we find

$$\begin{aligned} \tilde{\Delta} &= n \cdot \tilde{\Delta}\{n=1\} \\ \bar{\Delta} &= \binom{n}{2} \bar{\Delta}\{n=2\} \end{aligned} \tag{C.7}$$

We begin by calculating $\tilde{\Delta}\{n=1\}$.

The H_4 term in $\tilde{\Delta}$ will involve terms in the quartic action with two Y's replaced by J's, while the H_3H_3 term in $\tilde{\Delta}$ will involve terms in the cubic action with one Y replaced by a J. The calculation therefore reduces to evaluating correlation functions involving the following operators which are quadratic in the fields.

$$\begin{aligned} A &= \mathrm{Tr}(i\epsilon^{ijk} J^i [Y^j, Y^k])\,, \\ B &= \mathrm{Tr}([J^i, Y^j]^2)\,, \\ C &= \mathrm{Tr}([J^i, X^a]^2)\,, \\ D &= \mathrm{Tr}(\psi^\dagger \sigma^i [J^i, \psi] + \psi \sigma^i [J^i, \psi^\dagger])\,. \end{aligned}$$

In terms of the normalized coordinates, these are given by

$$\begin{aligned} A &= \sum (j-1) \beta^\dagger_{jm}\beta_{jm} - \sum (j+2)\alpha^\dagger_{jm}\alpha_{jm}\,, \\ B &= -\sum j(j-1)\beta^\dagger_{jm}\beta_{jm} - \sum (j+1)(j+2)\alpha^\dagger_{jm}\alpha_{jm}\,, \\ C &= -\sum j(j+1)(x^a_{jm})^\dagger x^a_{jm}\,, \\ D &= \sum (j-\frac{1}{2})(\eta^\dagger_{jm}\eta_{jm} - \eta_{jm}\eta^\dagger_{jm}) + \sum \left(j+\frac{3}{2}\right)(\chi^\dagger_{jm}\chi_{jm} - \chi_{jm}\chi^\dagger_{jm})\,. \end{aligned}$$

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The expectation values of these operators may be calculated to be

$$\begin{aligned}
\langle A\rangle &= \frac{3(N-1)}{2N}\,,\\
\langle B\rangle &= -\frac{1}{2}(N-1)(4N^2+4N-3)\,,\\
\langle C\rangle &= -6N(N^2-1)\,,\\
\langle D\rangle &= -\frac{16}{3}N(N^2-1)
\end{aligned}$$

using the propagators

$$\begin{aligned}
\langle (x^a_{jm})^\dagger x^b_{\tilde{j}\tilde{m}}\rangle &= \delta^{ab}\delta_{j,\tilde{j}}\delta_{m,\tilde{m}}\frac{3}{2j+1}\,, \qquad \langle (\alpha_{jm})^\dagger \alpha_{\tilde{j}\tilde{m}}\rangle = \delta_{j,\tilde{j}}\delta_{m,\tilde{m}}\frac{3}{2(j+1)}\,,\\
\langle (\beta_{jm})^\dagger \beta_{\tilde{j}\tilde{m}}\rangle &= \delta_{j,\tilde{j}}\delta_{m,\tilde{m}}\frac{3}{2j}\,.
\end{aligned}$$

The H_4 contribution may now be quickly evaluated. We find

$$\begin{aligned}
\tilde{\Delta}_4^{XYXY} = \left\langle a\left(-\frac{1}{2}\,\mathrm{Tr}([X^a,Y^i]^2)\right)a^\dagger\right\rangle &= -\frac{1}{2}\cdot 2\cdot\frac{6}{N(N^2-1)}\langle C\rangle\\
&= 36
\end{aligned}$$

while

$$\begin{aligned}
\tilde{\Delta}_4^{YYYY} = \left\langle a\left(-\frac{1}{4}\,\mathrm{Tr}([Y^i,Y^j]^2)\right)a^\dagger\right\rangle &= -\frac{1}{4}\cdot\frac{6}{N(N^2-1)}(8\langle A\rangle+4\langle B\rangle)\\
&= 12-9\frac{N+2}{N^2(N+1)}\,.
\end{aligned}$$

Note that in the first step we have used the fact that

$$\mathrm{Tr}([J^i,Y^j][Y^i,J^j]) = \mathrm{Tr}([J^i,Y^i]^2)+\mathrm{Tr}([J^i,J^j][Y^i,Y^j])$$

but the first term on the right involves only the gauge degrees of freedom, so we may set it to zero.

We next calculate

$$\tilde{\Delta}_3 = -\sum_i\frac{1}{E_i}\langle 0|a(H_3^i)^\dagger H_3^i a^\dagger|0\rangle\,.$$

The connected contributions are given by

$$\begin{aligned}
\tilde{\Delta}_3 = &-\sum_i\frac{1}{E_i}\langle 0|[[a,(H_3^i)^\dagger],a^\dagger]H_3^i|0\rangle-\sum_i\frac{1}{E_i}\langle 0|(H_3^i)^\dagger[[a,H_3^i],a^\dagger]|0\rangle-\\
&-\sum_i\frac{1}{E_i}\langle 0|[a,(H_3^i)^\dagger][H_3^i,a^\dagger]|0\rangle-\sum_i\frac{1}{E_i}\langle 0|[(H_3^i)^\dagger,a^\dagger][a,H_3^i]|0\rangle \qquad \text{(C.8)}
\end{aligned}$$

To calculate the terms on the second line, note that since H_3 depends on a and $a^\dagger$ only in the combination $(a+a^\dagger)$, we have

$$[H_3,a^\dagger] = [a,H_3]\equiv\sum\mathcal{O}_{E_i}\,,$$

where we have expanded the left hand expressions into a set of terms with distinct H_2 eigenvalues. Then

$$[a, H_3^i] = \mathcal{O}_{E_i - E_a}\,, \qquad [H_3^i, a^\dagger] = \mathcal{O}_{E_i + E_\alpha}\,.$$

Using these expressions, we have

$$\begin{aligned}\Delta_3 &= -\sum_i \frac{1}{E_i}\langle 0|\mathcal{O}^\dagger_{E_i+E_\alpha}\mathcal{O}_{E_i+E_\alpha}|0\rangle - \sum_i \frac{1}{E_i}\langle 0|\mathcal{O}^\dagger_{E_i-E_\alpha}\mathcal{O}_{E_i-E_\alpha}|0\rangle \\ &= -\sum_i \frac{2E_i}{E_i^2 - E_a^2}\langle 0|\mathcal{O}^\dagger_{E_i}\mathcal{O}_{E_i}|0\rangle\,,\end{aligned}$$

where $E_a = 1/3$ is the energy of the oscillator under consideration.

To proceed, we note first that all disconnected terms in the expression above vanish since these will have $E^i = 0$. To calculate the connected terms, we may use the explicit expressions for the operators $\mathcal{O}$ obtained from

$$\begin{aligned}[H_3, a^\dagger] &= \sqrt{\frac{6}{N(N^2-1)}}\left(-\frac{1}{6}A - \frac{1}{3}B - \frac{1}{3}C + \frac{1}{2}D\right) \\ &= \sqrt{\frac{6}{N(N^2-1)}}\left(\frac{1}{6}(j-1)(2j-1)\beta_{jm}\beta^\dagger_{jm} + \frac{1}{6}(j+2)(2j+3)\alpha_{jm}\alpha^\dagger_{jm} + \right.\\ &\qquad + \frac{1}{3}j(j+1)x^a_{jm}(x^a_{jm})^\dagger + \frac{1}{2}\left(j - \frac{1}{2}\right)(\eta^\dagger_{jm}\eta_{jm} - \eta_{jm}\eta^\dagger_{jm}) + \\ &\qquad \left. + \frac{1}{2}\left(j + \frac{3}{2}\right)(\chi^\dagger_{jm}\chi_{jm} - \chi_{jm}\chi^\dagger_{jm})\right).\end{aligned}$$

Note that the fermions terms contain only pieces with $E_0 = 0$ and therefore will not contribute. The α, β, and x contributions may be calculated separately using the propagators given above. For example, the β term will be

$$\begin{aligned}&\frac{6}{N(N^2-1)}\frac{-2(2j/3)}{(2j/3)^2 - (1/3)^2}\left\langle \frac{1}{6}(j-1)(2j-1)\beta_{jm}\beta^\dagger_{jm}\;\frac{1}{6}(j-1)(2j-1)\beta_{jm}\beta^\dagger_{jm}\right\rangle = \\ &\quad = \frac{6}{N(N^2-1)}\sum_{j=1}^{N}\frac{-2(2j/3)}{(2j/3)^2 - (1/3)^2}\cdot\left(\frac{1}{6}(j-1)(2j-1)\right)^2\cdot 2\cdot\left(\frac{3}{2j}\right)^2(2j+1) \\ &\quad = -\frac{9}{N(N^2-1)}\sum_{j=1}^{N}\frac{(j-1)^2(2j-1)}{j}\,,\end{aligned}$$

where in the second line the third factor is from the two ways of contracting the βs, the fourth factor is from the propagators, and the final factor is from the sum over m.

Similarly, the α contribution gives

$$-\frac{9}{N(N^2-1)}\sum_{j=0}^{N-2}\frac{(j+2)^2(2j+3)}{(j+1)}\,.$$

Together, these sums may be evaluated to give

$$\tilde{\Delta}_3^{YYY} = -12 - 9\frac{5N+1}{N^2(N+1)}\,.$$

The contribution from the x terms is

$$\tilde{\Delta}_3^{XXY} = -\frac{108}{N(N^2-1)}\sum_{j=0}^{N-1} j(j+1) = -36\,.$$

The remaining contribution to $\tilde{\Delta}_3$ comes from the terms on the first line of equation (C.8), which we denote by $\tilde{\Delta}_3'$. To compute these, we use

$$[[a,H_3],a^\dagger] = 3\sqrt{\frac{6}{N(N^2-1)}}(a+a^\dagger)\,. \tag{C.9}$$

Then the first line of equation (C.8) becomes

$$\begin{aligned}
\tilde{\Delta}_3' &= -3\cdot 3\sqrt{\frac{6}{N(N^2-1)}}(\langle 0|aH_3|0\rangle + \langle 0|H^3 a^\dagger|0\rangle) \\
&= -\frac{108}{N(N^2-1)}\left\langle -\frac{1}{6}A - \frac{1}{3}B - \frac{1}{3}C + \frac{1}{2}D\right\rangle \\
&= \frac{27(2N+1)}{N^2(N+1)}
\end{aligned}$$

Combining all the terms in $\tilde{\Delta}$, we see that $\tilde{\Delta}_3^{XXY}$ cancels $\tilde{\Delta}_4^{XYXY}$, while $\tilde{\Delta}_4^{YYYY}$, $\tilde{\Delta}_3^{YYY}$, and $\tilde{\Delta}_3'$ sum to zero, so that

$$\tilde{\Delta} = 0$$

We now proceed to calculate $\bar{\Delta}\{n=2\}$. In this case, the H_4 contribution involves only the Y^4 terms with each Y contracted with and a or $a^\dagger$. Explicitly, we find

$$\begin{aligned}
\bar{\Delta}_4 &= \frac{1}{2}\left\langle aa\,\mathrm{tr}\left(-\frac{1}{4}[Y^i,Y^j][Y^i,Y^j]\right) a^\dagger a^\dagger|0\right\rangle_* \\
&= -\frac{1}{2}\frac{1}{4}\left[\frac{6}{N(N^2-1)}\right]^2\ 24\,\mathrm{tr}([J^i,J^j]^2) \\
&= \frac{54}{N(N^2-1)}\,,
\end{aligned}$$

where the $\langle\,\cdot\,\rangle_*$ indicates that we are keeping only terms with no contractions between initial and final as. To compute the H_3H_3 terms, we note that

$$H_3 = \frac{2}{3\sqrt{N(N^2-1)}}\alpha_{00}^3 + \cdots\,,$$

where the additional terms have no more than one power of α_{00} and therefore cannot contribute to $\bar{\Delta}$. Then

$$\bar{\Delta}_3 = \sum -\frac{1}{E_i}\langle\psi|(H_3^i)^\dagger H_3^i|\psi\rangle_* \tag{C.10}$$

$$= -\sum\langle aa\alpha_{00}^3\alpha_{00}^3 a^\dagger a^\dagger\rangle\frac{4}{9N(N^2-1)}\frac{1}{2} \tag{C.11}$$

$$= \frac{1}{2}\frac{4}{9N(N^2-1)}\left\{\left(\sqrt{\frac{3}{2}}\right)^4\frac{3}{2}\right\}(3\cdot[36] - 1\cdot[36] - 3\cdot[144]) \tag{C.12}$$

$$= \frac{-270}{N(N^2-1)}\,, \tag{C.13}$$

where in the last line, terms in { } brackets are from propagators, and the terms in [ ] brackets are combinatorical factors for the contributions with $E_i = -1/3$, $E_i = 1/3$, and $E_i = 1$ respectively.

Thus, for $n = 1$ the total energy shift is zero and for $n = 2$ we find

$$\begin{aligned} \bar{\Delta} &= \bar{\Delta}_3 + \bar{\Delta}_4 \\ &= -\frac{216}{N(N^1 - 1)} \,. \end{aligned}$$

We may now use (C.7) to write the result for the leading energy shift of the breathing mode with occupation number n,

$$\begin{aligned} \Delta &= -216 \binom{n}{2} \frac{R^3}{\mu^3 N(N^2 - 1)} \mu \\ &= -216 \binom{n}{2} g^2 \mu \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \end{aligned}$$

Thus, for the low-energy states, the leading perturbative correction is proportional to the bare coupling $g^2 = \{R/(\mu N)\}^3$, which remains finite in the large-N limit.

D. Height of the potential

In this appendix, we will demonstrate explicitly the existence of a path between the irreducible vacuum (labeled by $N + 1$) and the vacuum labeled by $(N, 1)$ along which the maximum energy is of order $(\mu N/R)^3 1/N = 1/g^2 N$ rather than $1/g^2$.[19]

Our choice of path is physically motivated by the following picture. The initial vacuum state corresponds to a single membrane sphere of radius $\mu N/6R$, while the final state corresponds to a slightly smaller sphere of radius $\mu(N-1)/6R$ plus a small sphere of membrane at the origin. If these were really continuum membranes, one way to move between these two points continuously at a seemingly small cost in energy would be to form a thin tube of membrane from the south pole of the original sphere to the origin where the small second sphere could grow from the tip of the tube. In this way, some of the membrane from the outer sphere could be transferred to the origin, yielding the desired final configuration, as shown in figure 2.

We can translate this continuum picture to a path between the desired initial and final matrix configurations as follows. First, write down a map from the unit sphere (the membrane worldvolume) to the target space which is a function of some parameter ϵ which will be related to the radius of the inner sphere. At a given value of ϵ there will be some line of latitude near the south pole of the worldvolume sphere across which there is a singularity corresponding to the infinitesimal tube. The area above this line will map to the outer sphere, while the (small) area below this line will map to the small inner sphere.[20]

[19] In terms of the natural metric $\mathrm{Tr}((X^A)^2)$ on the space of matrices, this choice for the final state represents the closest vacuum.

[20] We will not need to explicitly specify the mapping, but a natural choice would be to take the sum of the radii of the target space spheres to be a constant, and impose the condition that the map preserves ratios of areas.

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Once we have defined such a continuum mapping, we may obtain a corresponding path in the space of matrices by expanding the continuum map in terms of spherical harmonics,

$$x^i(\theta, \psi, \epsilon) = \sum x^i_{lm}(\epsilon) Y_{lm}(\theta, \phi)$$

and then replacing the spherical harmonics with matrix spherical harmonics with $l < N$,

$$X^i(\epsilon) = \sum_{l<N} x^i_{lm}(\epsilon) Y^{N\times N}_{lm} \,.$$

Without explicitly specifying a map, it is clear that a natural choice is to preserve axial symmetry at all points along the path, so that we may choose x^3 to be a function of θ only and $x^1 + ix^2 = r(\theta)e^{i\phi}$. In this case, the expansion of $x^3(\theta, \phi)$ will involve only spherical harmonics with $m = 0$, while the expansion of $x^1 + ix^2$ will involve only spherical harmonics with $m = 1$. As a result, at every point along the path, our matrices will take the form

$$X^3 = \frac{\mu}{3R}\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N+1} \end{pmatrix}, \qquad X^+ = \frac{\mu}{3R}\begin{pmatrix} 0 & b_1 & & \\ & 0 & \ddots & \\ & & \ddots & b_N \\ & & & 0 \end{pmatrix}$$

since the matrices Y_{l0} are all diagonal, while the matrices Y_{l1} have non-zero entries only in the first row above the diagonal in each column.

It is convenient to write the potential explicitly in terms of these variables. We find

$$V = \frac{1}{2}\mu\left(\frac{\mu}{3R}\right)^4 \left(\sum_{l=1}^{N+1}\left(a_l + \frac{1}{2}(b^2_{l-1} - b^2_l)\right)^2 + \sum_{l=1}^{N} b^2_l (a_{l+1} - a_l + 1)^2\right). \tag{D.1}$$

Since this expression is quadratic in the variables b^2_l, it is not hard to determine the set of b's for which the potential is minimum for a given set of a's. The extremum condition for b_l is

$$2b_l\left(\left(a_{i+1} - a_i + \frac{3}{2}\right)^2 - \frac{5}{4} + b^2_l - \frac{1}{2}(b^2_{l+1} + b^2_{l-1})\right) = 0\,.$$

Thus, at the minimum of the potential for a fixed set of a's, we will have for each l either $b_l{=}0$ or

$$b^2_{l+1} - 2b^2_l + b^2_{l-1} = 2\left(a_{l+1} - a_l + \frac{3}{2}\right)^2 - \frac{5}{2}\,. \tag{D.2}$$

Our problem is now reduced to specifying $a_l(\epsilon)$ such that the initial configuration is

$$a_l(0) = \left(\frac{N}{2}, \frac{N}{2} - 1, \ldots, -\frac{N}{2} + 1, -\frac{N}{2}\right)$$

and the final configuration is

$$a_l(1) = \left(\frac{N-1}{2}, \frac{N-3}{2}, \ldots, -\frac{N-1}{2}, 0\right).$$

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In principle, one could let the value of a_N parameterize the path and explicitly extremize the potential in terms of the other a_i's to determine the lowest energy path satisfying our ansatz of axial symmetry. However, since the values of a_l for $l \leq N$ change very little between their initial and final values it seems likely that this minimum path will have a maximum energy that does not differ significantly from that of the linear path between initial and final points. In other words, we expect that working to linear order in ϵ will be sufficient to determine the order of magnitude of the barrier height.

Thus, we take

$$a_l = \frac{N}{2} + 1 - l - \frac{\epsilon}{2}, \qquad a_{N+1} = -\frac{N}{2} + \frac{N}{2}\epsilon .$$

At $\epsilon = 0$, the values of b correspond to $X^+ = \mu/3RJ^+$,

$$b_k^2 = k(N+1-k)$$

so near $\epsilon = 0$ we will be on the branch with all b_i's non zero. Inserting the values of a into (D.2), we find

$$\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & & \\ & & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} \begin{pmatrix} b_1^2 \\ \vdots \\ b_{N-1}^2 \\ b_N^2 \end{pmatrix} = \begin{pmatrix} -2 \\ \vdots \\ -2 \\ -2 + \frac{1}{2}(N+1)^2\epsilon^2 + (N+1)\epsilon \end{pmatrix} .$$

The matrix M on the left may be inverted to give a symmetric matrix with

$$M^{-1}_{kl} = -k\frac{N+1-l}{N+1}, \qquad k \leq l$$

so we find that near $\epsilon = 0$

$$b_k^2 = k\left(N + 1 - k - \epsilon - \frac{1}{2}(N+1)\epsilon^2\right) .$$

Note that b_N in this formula is negative for $\epsilon > (\sqrt{2N+3}-1)/(N+1)$, so for these values of ϵ, we are no longer on the branch with all b's non zero. In fact, at $\epsilon = (\sqrt{2N+3}-1)/(N+1)$ the values of b,

$$b_k^2 = k(N-k)$$

are already equal to their $\epsilon = 1$ values corresponding to J^+ in the $(N, 1)$ representation. It is not hard to check that these values, together with the values of a defined above, satisfy eq. (D.2) for all values of ϵ. Thus, we may conclude that

$$b_k^2 = \begin{cases} k(N + 1 - k - \epsilon - \frac{1}{2}(N+1)\epsilon^2) , & \epsilon \leq \frac{\sqrt{2N+3}-1}{N+1} , \\ k(N-k) , & \epsilon \geq \frac{\sqrt{2N+3}-1}{N+1} . \end{cases}$$

Plugging the values for a and b into (D.1) we conclude that the energy along our path is given by

$$V = \begin{cases} \mu\left(\frac{\mu N}{R}\right)^3 \frac{1}{648}\left(\frac{(N+1)\epsilon}{N}\right)^2 (1 - \epsilon - \frac{1}{4}(N+1)\epsilon^2) , & \epsilon \leq \frac{\sqrt{2N+3}-1}{N+1} , \\ \mu\left(\frac{\mu N}{R}\right)^3 \frac{1}{648}\frac{(N+1)}{N^2}(1-\epsilon)^2 , & \epsilon \geq \frac{\sqrt{2N+3}-1}{N+1} . \end{cases}$$

From this expression, we find that the maximum of the potential along the path we have determined is

$$\frac{V_{\max}}{\mu} = \left(\frac{\mu N}{R}\right)^3 \left\{\frac{1}{648N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right)\right\}.$$

This represents an upper bound on the classical energy required to move between the vacuum labeled by $\{N+1\}$ and the vacuum labeled by $\{N, 1\}$. It is clear that the same bound applies for a path between vacua $\{N+1, N_1, \ldots, N_n\}$ and $\{1, N, N_1, \ldots, N_n\}$. Thus, the minimum energy required to move from a given vacuum $\{N_1, N_2, \ldots, N_n\}$ to another vacuum is not more than $1/N_{\min}(\mu N_{\min}/R)^3$, where $N_{\min}$ corresponds to the smallest sphere in the collection.

Finally, we note that by a series of these steps in which we move from a vacuum $\{N_1, N_2, \ldots, N_n\}$ to a vacuum $\{1, N_1 - 1, N_2, \ldots, N_n\}$, and then by the reverse process to the vacuum $\{N_1, N_2 + 1, \ldots, N_n\}$, we may transfer any number of momentum units between any two membrane spheres and in this way move between any two vacua of the theory. The maximum energy required for a given initial and final vacuum will be of order

$$E \sim \frac{1}{N_{\max}} \left(\frac{\mu N_{\max}}{R}\right)^3,$$

where $N_{\max}$ corresponds to the largest sphere whose momentum is changed in the process.

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